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Domination Equitable Coloring of graphs

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Abstract

A domination coloring of a graph G is a proper vertex coloring, where each vertex of G dominates at least one color class, and each color class is dominated by at least one vertex. The minimum number of colors among all domination coloring sets of vertices is called the domination chromatic number. In this paper, we introduce equitability in domination coloring and define a new variant of proper coloring of graphs. We obtain domination equitable chromatic number of standard graphs namely path, cycle and star.

1 Introduction

Graph coloring, a cornerstone of graph theory, entails assigning colors to vertices to prevent adjacent vertices from sharing the same color. Domination equitable coloring introduced in this paper extends the intertwining

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principles of coloring, domination and equitability [1]. In our daily lives, we encounter situations when we need to partition a system with binary conflict relations into subsystems that are either equal or nearly equal in terms of conflict resolution. In such instances, we may represent this scenario using equitable graph colouring. Recent strides, notably by Chellali and Volkmann [2, 3], unveil connections between graph chromatic number and domination parameters. Gera et al.'s [4] dominator coloring concept, scrutinized by Arumugam et al. [5] and Gera [6] sparked extensive research. Walter Meyer introduced first the notion of equitability in 1973.
We begin with a formal definition of domination coloring.

Definition 1.1. *A domination coloring of a graph G is a proper vertex coloring of G such that each vertex of G dominates at least one color class, and each color class is dominated by at least one vertex. The minimum number of colors among all domination colorings is called the domination chromatic number, denoted by $\chi_{dd}(G)$.*

Definition 1.2. *A proper coloring of a graph G is said to be equitably $k-$colorable if the number of vertices of any two color classes $C_1, C_2, \ldots, C_k$ of a graph G differ by at most one. That is, $||C_i| - |C_j|| \leq 1$ for $1 \leq i, j \leq k$. The chromatic number of an equitable coloring is denoted by $\chi_e(G)$.*

We introduce equitability in domination coloring of graphs and define *Domination Equitable Coloring* as a new variant of coloring concept in graphs.

Definition 1.3. *Let $G = (V, E)$ be a graph. A domination equitable coloring of G is a proper vertex coloring that satisfies the following conditions:*
(i) Each vertex v dominates at least one color class.
(ii) Each color class is dominated by at least one vertex.
(iii) If G has k color classes $C_1, C_2, C_3, \ldots, C_k$, then the number of vertices in any two color classes C_i and C_j differ by at most one: $||C_i| - |C_j|| \leq 1$ for every $1 \leq i, j \leq k$. The minimum number of colors required for a domination equitable coloring of G is denoted by $\chi_{dde}(G)$ and is referred to as the domination equitable chromatic number of G.

2 Main results

Lemma 2.1. *Let P_n be a path of length n. Then $\chi_{dde}(P_n) \geq 2\lfloor \frac{n}{3} \rfloor$.*

Proof: Let the vertices of P_n be sequentially labelled as $v_1, v_2, ..., v_n$ from left to right. See Figure 1(a). Let v_1 be in color class C_1 and v_2 be in color

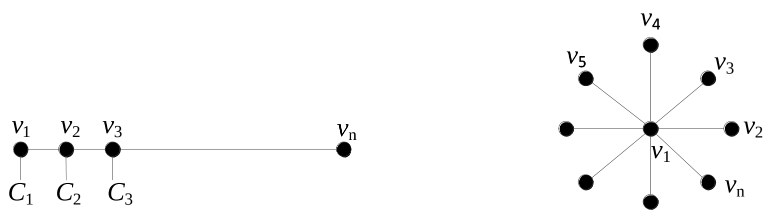

Figure 1: (a) PathP_n; (b) Star $K_{1,n-1}$

class C_2. By definition of domination coloring v_1 dominates a color class and since it is of degree 1, none of the vertices $v_3, v_4, ...$ or v_n can be in color class C_2. If v_3 is in color class C_1, then v_2 dominates color class C_1. Since degree of v_2 is 2, no vertex $v_4, v_5, ...$ or v_n can be in color class C_1. Thus two color classes are necessary for the subpath $v_1v_2v_3$. This is true for every successive subpath on 3 vertices. Thus $\chi_{dde}(P_n) \geq 2\lfloor \frac{n}{3} \rfloor$.
Remark: Let C_n be a cycle of length n. Proceeding as in Lemma 2.1, we have $\chi_{dde}(C_n) \geq 2\lfloor \frac{n}{3} \rfloor$.

Theorem 2.2. *Let P_n and C_n be a path and a cycle of length n respectively. Then $\chi_{dde}(P_n) = 2\lfloor \frac{n}{3} \rfloor + mod(n, 3), n \geq 2$ and*

$$\chi_{dde}(C_n) = \begin{cases} 2, & n = 4, \\ 3, & n = 3, 5, \\ 2\lfloor \frac{n}{3} \rfloor + mod(n, 3), & otherwise \end{cases}$$

Proof: Consider the path P_n on n vertices. Name the i^{th} vertex of P_n from the left as $v_i, 1 \leq i \leq n$. Let $D_i = \{v_{3i-2}, v_{3i-1}, v_{3i}\}, 1 \leq i \leq \lfloor \frac{n}{3} \rfloor$. Partition P_n into domination equitable color classes as follows:
(i) Let the mid vertex of D_i be in color class C_{2i-1} and the remaining two vertices of every D_i be in $C_{2i}, 1 \leq i \leq \lfloor \frac{n}{3} \rfloor$.
(ii) When $n \equiv 1 \pmod 3$, color the vertex v_n as $C_{2\lfloor \frac{n}{3} \rfloor + 1}$.
(iii) When $n \equiv 2 \pmod 3$, color the vertices v_{n-1} and v_n as $C_{2\lfloor \frac{n}{3} \rfloor + 1}$ and $C_{2\lfloor \frac{n}{3} \rfloor + 2}$ respectively.
Thus when $n \equiv 0 \pmod 3$, $\chi_{dde}(P_n) \leq 2\lfloor \frac{n}{3} \rfloor$. By Lemma 2.1, $\chi_{dde}(P_n) = 2\lfloor \frac{n}{3} \rfloor$. On the otherhand, when $n \equiv 1$ or $2 \pmod 3$, 1 or 2 new colors respectively are necessary for minimum domination equitable coloring. Hence the result.
It is easy to verify that $\chi_{dde}(C_4) = 2, \chi_{dde}(C_n) = 3$ when $n = 3$ or 5. When $n \geq 6$, the argument is similar to that of P_n.

Theorem 2.3. *Let G be a graph on n vertices with maximum degree $n - 1$. Then $\lfloor \frac{n}{2} \rfloor + 1 \leq \chi_{dde}(G) \leq n$*

Proof: Let v be a vertex of degree $n-1$ in G. Then v is in a color class of cardinality 1. Since the coloring is equitable, every other color class is of cardinality at most 2. Then $\chi_{dde}(G) \geq \frac{n-1}{2}+1$ when n is odd and $\chi_{dde}(G) \geq \frac{n-2}{2}+2$ when n is even. Thus, $\chi_{dde}(G) \geq \lfloor \frac{n}{2} \rfloor + 1$ for any n.

Corollary 2.4. *Let $K_{1,n-1}$ be a star graph on n vertices. Then* $\chi_{dde}(K_{1,n-1}) = \lfloor \frac{n}{2} \rfloor + 1, \forall n \geq 3$.

Proof: Name the vertex of degree $n-1$ as v_1 and all other vertices as $v_2, v_3, ..., v_n$ in the anti clockwise sense. See Figure 1(b). Let v_1 be in the single color class C_1. Since $v_2, v_3, ..., v_n$ are independent set of vertices and the coloring is equitable, pairs of vertices should get distributed in $\frac{n-1}{2}$ number of distinct color classes when n is odd and $\frac{n-2}{2}$ number of distinct color classes together with a single vertex in a unique color class when n is even. Thus, $\chi_{dde}(K_{1,n-1}) \leq \lfloor \frac{n}{2} \rfloor + 1$. By Theorem 2.3, $\chi_{dde}(K_{1,n-1}) = \lfloor \frac{n}{2} \rfloor + 1$.

3 Conclusion

In this paper, we introduce the concept of domination equitable coloring. Our future work will focus on determining domination equitable chromatic number for several interconnection networks such as hypercubes, butterfly networks and helms.

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