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Fundamental Properties of a Class of Analytic Functions Defined by a Generalized Multiplier Transformation Operator

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Abstract

In this paper, we introduce and geometrically investigate a new subclass $\mathcal{B}^{*}_{m,\delta}(\lambda,\mu;\alpha)$ of analytic functions defined by a generalized multiplier transformation operator. Moreover, our investigation includes coefficient estimates, growth and distortion theorems, and closure theorems relevant to these functions. Furthermore, we establish conditions under which functions in this subclass exhibit properties such as convexity, close-to-convexity, and starlikeness.

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1 Introduction

Let $\mathcal{A}$ represent the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Consider the subclass $\mathcal{A}^*$ of functions in $\mathcal{A}$ that are of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0. \tag{1.2}$$

For $f \in \mathcal{A}$, Cho and Srivastava [1] introduced the following generalized multiplier transformation operator

$$I_\delta^m f(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+\delta}{1+\delta}\right)^m a_n z^n, \quad (m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \text{ and } \ \delta \geq 0). \tag{1.3}$$

Note that, for $\delta = 1$, the multiplier transformation I_δ^m was introduced and studied by Uralegaddi and Somanatha [2] and for $\delta = 0$, the multiplier transformation I_δ^m was introduced and studied by Salagean [3].

Yousef et al. [4, 5] introduced the following general class $\mathcal{B}_\Sigma^\eta(\lambda, \mu; \alpha)$ of analytic and bi-univalent functions:

Definition 1.1. For $\lambda \geq 1, \eta \geq 0, \mu \geq 0$ and $0 \leq \alpha < 1$, a function$f \in \mathcal{A}$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma^\eta(\lambda, \mu; \alpha)$ if the following conditions hold for all $z \in \mathbb{U}$:

$$Re\left((1-\lambda)\left(\frac{f(z)}{z}\right)^\eta + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\eta-1} + \xi\mu z f''(z)\right) > \alpha, \tag{1.4}$$

where $\xi = \frac{2\lambda+\eta}{2\lambda+1}$.

Several researchers have explored the class $\mathcal{B}_\Sigma^\eta(\lambda, \mu; \alpha)$ and utilized it in various contexts. For example, we refer the reader to [6, 7, 8, 9, 10, 11, 12, 13, 14, 16].

When $I_\delta^m f(z)$ is incorporated into Definition 1.1 and letting $\eta = 1$, we introduce here a class $\mathcal{B}_{m,\delta}(\lambda, \mu; \alpha)$ defined as follows:

Definition 1.2. For $\lambda \geq 1, \mu \geq 0$ and $0 \leq \alpha < 1$, a function$f \in \mathcal{A}$ given by (1.1) is said to be in the class $\mathcal{B}_{m,\delta}(\lambda, \mu; \alpha)$ if the following conditions hold for all $z \in \mathbb{U}$:

$$\operatorname{Re}\left\{(1-\lambda)\frac{I_\delta^m f(z)}{z} + \lambda(I_\delta^m f(z))' + \mu z(I_\delta^m f(z))''\right\} > \alpha. \tag{1.5}$$

Moreover, let $\mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha) = \mathcal{B}_{m,\delta}(\lambda, \mu; \alpha) \cap \mathcal{A}^*$.

Inspired by the work of Amourah and Yousef [15], we investigate similar geometric properties for functions in the class $\mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha)$.

2 Coefficient estimates

This section commences with deriving a necessary and sufficient condition for a function $f(z)$ to belong to the class $\mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha)$.

Theorem 2.1. *A function $f \in \mathcal{A}^*$ defined by (1.2) belongs to the class $\mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha)$ if and only if*

$$\sum_{n=2}^{\infty}[(1-\lambda) + n\lambda + n(n-1)\mu]\left(\frac{n+\delta}{1+\delta}\right)^m a_n \leq 1-\alpha. \tag{2.1}$$

Proof. Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha)$ and $z \in \mathbb{U}$. Then by definition

$$\begin{aligned}&\operatorname{Re}\left\{(1-\lambda)\frac{I_\delta^m f(z)}{z} + \lambda(I_\delta^m f(z))' + \mu z(I_\delta^m f(z))''\right\}\\ &= \operatorname{Re}\left\{1 - \sum_{n=2}^{\infty}[(1-\lambda) + n\lambda + n(n-1)\mu]\left(\frac{n+\delta}{1+\delta}\right)^m a_n z^{n-1}\right\} > \alpha.\end{aligned}$$

As z approaches 1^- along the real axis, inequality (2.1) follows.

Conversely, if the inequality (2.1) holds, then for $z \in \mathbb{U}$, we have

$$\begin{aligned}&\left|(1-\lambda)\frac{I_\delta^m f(z)}{z} + \lambda\left(I_\delta^m f(z)\right)' + \mu z\left(I_\delta^m f(z)\right)'' - 1\right|\\ &= \left|\sum_{n=2}^{\infty}[(1-\lambda) + n\lambda + n(n-1)\mu]\left(\frac{n+\delta}{1+\delta}\right)^m a_n z^{n-1}\right|\\ &\leq \sum_{n=2}^{\infty}[(1-\lambda) + n\lambda + n(n-1)\mu]\left(\frac{n+\delta}{1+\delta}\right)^m a_n |z|^{n-1}\\ &\leq \sum_{n=2}^{\infty}[(1-\lambda) + n\lambda + n(n-1)\mu]\left(\frac{n+\delta}{1+\delta}\right)^m a_n \leq 1-\alpha.\end{aligned}$$

This yields $f \in \mathcal{B}^*_{m,\delta}(\lambda,\mu;\alpha)$. □

Corollary 2.2. Let $f(z)$ defined by (1.2) be in the class $\mathcal{B}^*_{m,\delta}(\lambda,\mu;\alpha)$. Then

$$a_n \leq \frac{1-\alpha}{[(1-\lambda)+n\lambda+n(n-1)\mu]\left(\frac{n+\delta}{1+\delta}\right)^m}, \quad n \geq 2. \tag{2.2}$$

The equality in (2.2) holds for the function

$$f(z) = z - \frac{1-\alpha}{[(1-\lambda)+n\lambda+n(n-1)\mu]\left(\frac{n+\delta}{1+\delta}\right)^m} z^n.$$

3 Growth and distortion theorems

In this section, we establish the growth and distortion theorems for any function $f(z)$ belonging to the class $\mathcal{B}^*_{m,\delta}(\lambda,\mu;\alpha)$.

Theorem 3.1. *Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{B}^*_{m,\delta}(\lambda,\mu;\alpha)$. Then, for $|z| = r < 1$, we have*

$$r - \frac{1-\alpha}{(1+\lambda+2\mu)\left(\frac{2+\delta}{1+\delta}\right)^m} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{(1+\lambda+2\mu)\left(\frac{2+\delta}{1+\delta}\right)^m} r^2. \tag{3.1}$$

The equality in (3.1) holds for the function

$$f(z) = z - \frac{1-\alpha}{(1+\lambda+2\mu)\left(\frac{2+\delta}{1+\delta}\right)^m} z^2.$$

Proof. If the function $f(z) \in \mathcal{B}^*_{m,\delta}(\lambda,\mu;\alpha)$ defined by (1.2), then for $|z| = r < 1$, we have $|f(z)| \leq |z| + \sum_{n=2}^{\infty} a_n |z|^n \leq r + r^2 \sum_{n=2}^{\infty} a_n$.

By making use of Theorem 2.1, we get

$$|f(z)| \leq r + \frac{1-\alpha}{(1+\lambda+2\mu)\left(\frac{2+\delta}{1+\delta}\right)^m} r^2.$$

Similarly,

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} a_n |z|^{n-1} \geq r - \frac{1-\alpha}{(1+\lambda+2\mu)\left(\frac{2+\delta}{1+\delta}\right)^m} r^2.$$

Thus (3.1) holds which completes the proof of Theorem 3.1. □

Theorem 3.2. *Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{B}^*_{m,\delta}(\lambda,\mu;\alpha)$. Then, for $|z| = r < 1$, we have*

$$1 - \frac{2(1-\alpha)}{(1+\lambda+2\mu)\left(\frac{2+\delta}{1+\delta}\right)^m} r \le |f'(z)| \le 1 + \frac{2(1-\alpha)}{(1+\lambda+2\mu)\left(\frac{2+\delta}{1+\delta}\right)^m} r. \tag{3.2}$$

The equality in (3.2) holds for the function

$$f(z) = z - \frac{1-\alpha}{(1+\lambda+2\mu)\left(\frac{2+\delta}{1+\delta}\right)^m} z^2.$$

Proof. Apply a similar argument to the proof of Theorem 3.1 when taking the derivative of the function f. □

4 Closure theorems

Theorem 4.1. *Let $f_j(z) = z - \sum_{n=2}^{\infty} a_{nj} z^n$, $a_{nj} \ge 0$ and $z \in \mathbb{U}$, be in the class $\mathcal{B}^*_{m,\delta}(\lambda,\mu;\alpha)$ for $j = 1, 2, ..., J$. Then the function $F(z) = z - \sum_{n=2}^{\infty} b_n z^n$ is also in $\mathcal{B}^*_{m,\delta}(\lambda,\mu;\alpha)$, where $b_n = \frac{1}{J}\sum_{j=1}^{I} a_{nj}$, $\quad (n \ge 2)$.*

Proof. Let $f_j(z)$ be in $\mathcal{B}^*_{m,\delta}(\lambda,\mu;\alpha)$. Then

$$\begin{aligned}
&\sum_{n=2}^{\infty}[(1-\lambda)+n\lambda+n(n-1)\mu]\left(\frac{n+\delta}{1+\delta}\right)^m b_n \\
&= \sum_{n=2}^{\infty}[(1-\lambda)+n\lambda+n(n-1)\mu]\left(\frac{n+\delta}{1+\delta}\right)^m \left(\frac{1}{J}\sum_{j=1}^{J} a_{nj}\right) \\
&= \frac{1}{J}\sum_{j=1}^{J}\left(\sum_{n=2}^{\infty}[(1-\lambda)+n\lambda+n(n-1)\mu]\left(\frac{n+\delta}{1+\delta}\right)^m a_{nj}\right) \\
&\le \frac{1}{J}\sum_{j=1}^{J}(1-\alpha) = 1-\alpha,
\end{aligned}$$

where in the last inequality we have used Theorem 2.1.
This yields $F \in \mathcal{B}^*_{m,\delta}(\lambda,\mu;\alpha)$. □

Theorem 4.2. *The class $\mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha)$ is a convex set.*

Proof. Let $f_j(z) = z - \sum_{n=2}^{\infty} a_{nj} z^n$ be in the class $\mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha)$ for $j = 1, 2$.

We need to show that

$$\begin{aligned} G(z) &= \omega f_1(z) + (1-\omega) f_2(z) \\ &= z - \sum_{n=2}^{\infty} \left[\omega a_{n1} + (1-\omega) a_{n2}\right] z^n, \quad (0 \leq \omega \leq 1) \end{aligned}$$

is also in the class $\mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha)$.

Now, for $0 \leq \omega \leq 1$,

$$\begin{aligned} &\sum_{n=2}^{\infty} [(1-\lambda) + n\lambda + n(n-1)\mu] \left(\frac{n+\delta}{1+\delta}\right)^m [\omega a_{n1} + (1-\omega) a_{n2}] \\ &\quad = \omega \sum_{n=2}^{\infty} [(1-\lambda) + n\lambda + n(n-1)\mu] \left(\frac{n+\delta}{1+\delta}\right)^m a_{n1} \\ &\quad + (1-\omega) \sum_{n=2}^{\infty} [(1-\lambda) + n\lambda + n(n-1)\mu] \left(\frac{n+\delta}{1+\delta}\right)^m a_{n2} \\ &\quad \leq \omega(1-\alpha) + (1-\omega)(1-\alpha) = 1-\alpha. \end{aligned}$$

Hence $G(z) \in \mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha)$ which completes the proof of Theorem 4.2. □

In the next section, we shed light on the radii of close-to-convexity, starlikeness, and convexity for a function in the class $\mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha)$.

5 Radii of close-to-convexity, starlikeness, and convexity

Significant and extensively studied subclasses of the analytic function class $\mathcal{A}$ include the class $\mathcal{C}(\beta)$ of close-to-convex functions of order β in $\mathbb{U}$, the class $\mathcal{S}^*(\beta)$ of starlike functions of order β in $\mathbb{U}$, and the class $\mathcal{K}(\alpha)$ of convex functions of order β in $\mathbb{U}$. For all $z \in \mathbb{U}$ and some $0 \leq \beta < 1$, by definition, we have

$$\mathcal{C}(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left\{ f'(z) \right\} > \beta \right\}, \tag{5.1}$$

$$\mathcal{S}^*(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left\{ \frac{z f'(z)}{f(z)} \right\} > \beta \right\}, \tag{5.2}$$

and

$$\mathcal{K}(\beta) := \left\{ f \in \mathcal{A} : \mathrm{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta \right\}. \tag{5.3}$$

Next, we shall determine the radii of close-to-convexity, starlikeness, and convexity for functions $f \in \mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha)$.

Theorem 5.1. *If $f \in \mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha)$, then $f \in \mathcal{C}(\beta)$ in $\{z : |z| < r_1\}$, where*

$$r_1 = \inf_n \left\{ \frac{(1-\beta)[(1-\lambda) + n\lambda + n(n-1)\mu] \left(\frac{n+\delta}{1+\delta}\right)^m}{n(1-\alpha)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2.$$

Proof. Let $f \in \mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha)$.

$$|f'(z) - 1| = \left| \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} a_n n |z|^{n-1}.$$

Now, by using Theorem 2.1, $|f'(z) - 1| \leqslant 1 - \beta$, if

$$n|z|^{n-1} \leq \frac{(1-\beta)[(1-\lambda) + n\lambda + n(n-1)\mu] \left(\frac{n+\delta}{1+\delta}\right)^m}{(1-\alpha)}$$

or

$$|z| \leq \inf_n \left\{ \frac{(1-\beta)[(1-\lambda) + n\lambda + n(n-1)\mu] \left(\frac{n+\delta}{1+\delta}\right)^m}{n(1-\alpha)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2.$$

This completes the proof of Theorem 5.1. □

Theorem 5.2. *If $f \in \mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha)$, then $f \in \mathcal{S}^*(\beta)$ in $\{z : |z| < r_2\}$, where*

$$r_2 = \inf_n \left\{ \frac{(1-\beta)[(1-\lambda) + n\lambda + n(n-1)\mu] \left(\frac{n+\delta}{1+\delta}\right)^m}{(n-\beta)(1-\alpha)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2.$$

Proof. Let $f \in \mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha)$.

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{\sum_{n=2}^{\infty} (n-1) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}} \right| \leq \frac{\sum_{n=2}^{\infty} (n-1) a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}.$$

Now, by using Theorem 2.1, $\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \beta$, if

$$(n-\beta)|z|^{n-1} \leq \frac{(1-\beta)[(1-\lambda) + n\lambda + n(n-1)\mu] \left(\frac{n+\delta}{1+\delta}\right)^m}{(1-\alpha)},$$

or

$$|z| \leq \inf_{n} \left\{ \frac{(1-\beta)[(1-\lambda)+n\lambda+n(n-1)\mu]\left(\frac{n+\delta}{1+\delta}\right)^m}{(n-\beta)(1-\alpha)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2.$$

This completes the proof of Theorem 5.2. □

From definitions (5.2) and (5.3), it is easy to see that

$$f(z) \in \mathcal{K}(\alpha) \text{ iff } zf'(z) \in \mathcal{S}^*(\alpha).$$

Corollary 5.3. If $f \in \mathcal{B}^*_{m,\delta}(\lambda, \mu; \alpha)$, then $f \in \mathcal{K}(\beta)$ in $\{z : |z| < r_3\}$, where

$$r_3 = \inf_{n} \left\{ \frac{(1-\beta)[(1-\lambda)+n\lambda+n(n-1)\mu]\left(\frac{n+\delta}{1+\delta}\right)^m}{n(n-\beta)(1-\alpha)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2.$$

References

[1] N. E. Cho, H. M. Srivastava, Argument estimates of certain analytic functions defined by a class of multiplier transformations, Math. Comp. Modell., **37,** nos. 1-2, (2003), 39–49.

[2] B. A. Uralegaddi, C. Somanatha, Certain classes of univalent functions, Current topics in analytic function theory, (1992), 371–374.

[3] G. S. Salagean, Subclasses of univalent functions. In Complex Analysis—Fifth Romanian-Finnish Seminar: Part 1 Proceedings of the Seminar held in Bucharest in 1981, Springer: Berlin/Heidelberg, (2006), 362–372.

[4] F. Yousef, S. Alroud, M. Illafe, New subclasses of analytic and bi-univalent functions endowed with coefficient estimate problems, Analysis and Mathematical Physics, **11,** (2021), 1–12.

[5] F. Yousef, S. Alroud, M. Illafe, A comprehensive subclass of bi-univalent functions associated with Chebyshev polynomials of the second kind, Boletín de la Sociedad Matemática Mexicana, **26,** (2020), 329–339.

[6] S. Bulut, Coefficient estimates for a new general subclass of analytic bi-univalent functions, Kor. J. Math., **29,** no. 3, (2021), 519–526.

[7] M. Illafe, A. Amourah, M. Haji Mohd, Coefficient estimates and Fekete–Szegö functional inequalities for a certain subclass of analytic and bi-univalent functions, Axioms, **11,** no. 4, (2022), 147.

[8] A. Amourah, B. A. Frasin, M. Ahmad, F. Yousef, Exploiting the Pascal distribution series and Gegenbauer polynomials to construct and study a new subclass of analytic bi-univalent functions, Symmetry, **14,** no. 1, (2022), 147.

[9] F. Yousef, A. Amourah, B. A. Frasin, T. Bulboacă, An avant-garde construction for subclasses of analytic bi-univalent functions, Axioms **11,** no. 6, (2022), 267.

[10] M. Illafe, F. Yousef, M. Haji Mohd, S. Supramaniam, Initial coefficients estimates and Fekete–Szegö inequality problem for a general subclass of bi-univalent functions defined by subordination, Axioms, **12,** no. 3, (2023), 235.

[11] A. Hussen, M. Illafe, Coefficient bounds for a certain subclass of bi-univalent functions associated with Lucas-Balancing polynomials, Mathematics, **11,** no. 24, (2023), 4941.

[12] A. Amourah, A. Alsoboh, O. Ogilat, G. M. Gharib, R. Saadeh, M. Al Soudi, A generalization of Gegenbauer polynomials and bi-univalent functions. Axioms, **12,** no. 2, (2023), 128.

[13] T. Al-Hawary, A. Amourah, H. Almutairi, B. A. Frasin, Coefficient inequalities and Fekete–Szegö-type problems for family of bi-univalent functions. Symmetry, **15,** no. 9, (2023), 1747.

[14] A. Amourah, I. Aldawish, B. A. Frasin, T. Al-Hawary, Applications of shell-like curves connected with Fibonacci numbers, Axioms, **12,** no. 7, (2023), 639.

[15] A. A. Amourah, F. Yousef, Some properties of a class of analytic functions involving a new generalized differential operator, Bol. Soc. Paran. Mat, **38,** no. 6, (2020), 33–42.

[16] A. Hussen, A. Zeyani, Coefficients and Fekete-Szegö Functional Estimations of Bi-Univalent Subclasses Based on Gegenbauer Polynomials, Mathematics, **11,** no. 13, (2023), 2852.