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Evaluation of certain integrals in the Book of Gradshteyn and Ryzhik

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Abstract

In this paper, we address some improper integrals from the book by Gradshteyn and Ryzhik [4]. Our approach focuses on obtaining exact solutions involving only sums, avoiding the calculation of tedious derivatives as those appearing in the mentioned text.

1 Introduction.

The tables of series and integrals have been used over time. Among these, we mention [1], [2], [3]. After a search, we found that the table of integrals by Gradshteyn and Ryzhik [4] is the most popular among users of the scientific

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community.
On page 325, section 3.252, of [4], the following integral appears:

$$\int_0^\infty \frac{dx}{(ax^2+2bx+c)^n} = \frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial c^{n-1}}\left[\frac{1}{\sqrt{ac-b^2}}\operatorname{arccot}\frac{b}{\sqrt{ac-b^2}}\right], \tag{1.1}$$
$$\left[a>0, \quad ac>b^2\right].$$

If one needs to compute an integral like the one presented on the left-hand side of (1.1), for large values of n, applying formula (1.1) becomes cumbersome, as it involves calculating higher-order derivatives with respect to the variable c of a product where the arccotangent function appears. Therefore, it is necessary to find a solution that is more practical and explicit, one that does not involve the calculation of derivatives, as the one shown in *Proposition 1.* Additionally, in this work, we obtain similar formulas for the cases $ac < b^2$ and $ac = b^2$, which are not covered in [4].
Another integral found in section 3.252 page 325 is:

$$\begin{aligned}\int_0^\infty &\frac{xdx}{(ax^2+2bx+c)^n}\\ &= \frac{(-1)^n}{(n-1)!}\frac{\partial^{n-2}}{\partial c^{n-2}}\left\{\frac{1}{2(ac-b^2)} - \frac{b}{2(ac-b^2)^{\frac{3}{2}}}\operatorname{arccot}\frac{b}{\sqrt{ac-b^2}}\right\} \text{ for } ac>b^2\\ &= \frac{(-1)^n}{(n-1)!}\frac{\partial^{n-2}}{\partial c^{n-2}}\left\{\frac{1}{2(ac-b^2)} + \frac{b}{4(b^2-ac)^{\frac{3}{2}}}\ln\frac{b+\sqrt{b^2-ac}}{b-\sqrt{b^2-ac}}\right\} \text{ for } b^2>ac>0\\ &= \frac{a^{n-2}}{2(n-1)(2n-1)b^{2n-2}} \text{ for } ac=b^2.\end{aligned} \tag{1.2}$$

In this second integral, all three possible cases are considered, but it is evident that for the cases $ac > b^2$ and $b^2 > ac > 0$, the solution to the integral is not practical, especially when dealing with large values of n, as it involves calculating very high-order derivatives.
Finally, the following integral is presented in section 3.252 page 325:

$$\int_0^\infty \frac{dx}{(ax^2+2bx+c)^{n+\frac{3}{2}}} = \frac{(-2)^n}{(2n+1)!!}\frac{\partial^n}{\partial c^n}\left\{\frac{1}{\sqrt{c}(\sqrt{ac}+b)}\right\} \tag{1.3}$$
$$[a \geq 0, \quad c>0, \quad b>-\sqrt{ac}].$$

Taking the above into account, the aim of this study is to express each of the previous integrals by means of explicit formulas involving only sums, as shown in *propositions 2.1, 2.2, 2.3*, the central part of this work.

2 A reduction formula

In this section, we mention a well-known recurrence formula and we present a deduction of it here in order to maintain the self-contained nature of this article.
For $n \geq 1$ and $a > 0$ a real number, let J_n be the indefinite integral.

$$J_n = \int \frac{dx}{(ax^2 + 2bx + c)^n}, \tag{2.4}$$

completing the square, $ax^2 + 2bx + c = a(y^2 + \alpha^2)$, can be written as

$$y = x + \frac{b}{a}, \quad \alpha^2 = \frac{ac - b^2}{a^2}, \tag{2.5}$$

taking into account this change of variable in the integral given in (2.4), we have

$$a^n J_n = \int \frac{dy}{(y^2 + \alpha^2)^n}, \tag{2.6}$$

as

$$\int \frac{dy}{(y^2 + \alpha^2)^n} = \int \frac{y^2 + \alpha^2}{(y^2 + \alpha^2)^{n+1}} dy = \alpha^2 \int \frac{dy}{(y^2 + \alpha^2)^{n+1}} + \int \frac{y^2 dy}{(y^2 + \alpha^2)^{n+1}}, \tag{2.7}$$

then equality (2.6) can be written as

$$a^n J_n = \alpha^2 a^{n+1} J_{n+1} + \int \frac{y^2 dy}{(y^2 + \alpha^2)^{n+1}}, \tag{2.8}$$

the last integral on the right-hand side of (2.8) is evaluated using the method of integration by parts.
If

$$u = y, \quad dv = \frac{y dy}{(y^2 + \alpha^2)^{n+1}},$$

so,

$$du = dy, \quad v = \frac{-1}{2n} \frac{1}{(y^2 + \alpha^2)^n},$$

thus, we have:

$$\int \frac{y^2 dy}{(y^2 + \alpha^2)^{n+1}} = \frac{-y}{2n(y^2 + \alpha^2)^n} + \frac{1}{2n} \int \frac{dy}{(y^2 + \alpha^2)^n}, \tag{2.9}$$

taking into account (2.6), (2.8), and (2.9), we obtain

$$a^n J_n = \alpha^2 a^{n+1} J_{n+1} - \frac{y}{2n(y^2+\alpha^2)^n} + \frac{1}{2n} a^n J_n, \tag{2.10}$$

if we consider $\alpha \neq 0$, from equality (2.10), we isolate J_{n+1},

$$J_{n+1} = \frac{2n-1}{2n} \cdot \frac{1}{a} \cdot \frac{1}{\alpha^2} J_n + \frac{y}{2n\alpha^2 a^{n+1}} \cdot \frac{1}{(y^2+\alpha^2)^n},$$

Taking into account the expressions for y and α given in (2.5), J_{n+1} can be written as

$$J_{n+1} = \frac{(2n-1)a}{2n(ac-b^2)} J_n + \frac{ax+b}{2n(ac-b^2)} \cdot \frac{1}{(ax^2+2bx+c)^n}. \tag{2.11}$$

If we consider the sequences $\{c_n\}$ and $\{d_n\}$ whose general terms are given by:

$$c_n = \frac{(2n-1)a}{2n(ac-b^2)}, \quad d_n = \frac{ax+b}{2n(ac-b^2)} \cdot \frac{1}{(ax^2+2bx+c)^n},$$

and considering the formula (see [5]), finally from (2.11), we obtain

$$J_{n+1} = \frac{1}{4n}\binom{2n}{n} \frac{a^n}{(ac-b^2)^n} \left[J_1 + \sum_{k=1}^{n} \frac{2^{2k-1}(ac-b^2)^{k-1}}{a^k k \binom{2k}{k}} \cdot \frac{ax+b}{(ax^2+2bx+c)^k} \right], \tag{2.12}$$

where

$$J_1 = \begin{cases} \dfrac{1}{\sqrt{ac-b^2}} \arctan\left(\dfrac{ax+b}{\sqrt{ac-b^2}} \right), & \text{if } ac-b^2 > 0 \\ \dfrac{1}{2\sqrt{b^2-ac}} \log\left[\dfrac{ax+b-\sqrt{b^2-ac}}{ax+b+\sqrt{b^2-ac}} \right], & \text{if } ac-b^2 < 0. \end{cases}$$

Remark. When $\alpha = 0$; that is, if $ac - b^2 = 0$, from equality (2.10), we obtain an explicit expression for J_n:

$$J_n = -\frac{a^{n-1}}{2n-1} \cdot \frac{1}{(ax+b)^{2n-1}}, \quad \text{if } ac = b^2. \tag{2.13}$$

2.1 A look at the integral(1.1)

As our interest is focused on the study of certain improper integrals mentioned in the introduction, let us consider, for $n \in \mathbb{N}$, the improper integral I_{n+1}, given by:

$$I_{n+1} = \int_0^\infty \frac{dx}{(ax^2+2bx+c)^{n+1}}, \tag{2.14}$$

thus, from (2.12), (2.13), we are able to establish the following result:

Proposition 2.1.

$$I_{n+1} = \frac{1}{4^n}\binom{2n}{n}\frac{a^n}{(ac-b^2)^n}\left[I_1 - \frac{b}{2(ac-b^2)}\sum_{k=1}^{n}\left(\frac{4ac-4b^2}{ac}\right)^k \frac{(k-1)!\,k!}{(2k)!}\right], \tag{2.15}$$

where

$$I_1 = \begin{cases} \dfrac{1}{\sqrt{ac-b^2}}\left[\dfrac{\pi}{2} - \arctan\left(\dfrac{b}{\sqrt{ac-b^2}}\right)\right], & \text{if } ac-b^2>0, \\ \dfrac{1}{\sqrt{b^2-ac}}\log\left[\dfrac{b+\sqrt{b^2-ac}}{\sqrt{ac}}\right], & \text{if } ac-b^2<0, \end{cases} \tag{2.16}$$

finally, from (2.13), it follows that:

$$I_{n+1} = \frac{a^n}{(2n+1)b^{2n+1}}, \quad \text{if } ac=b^2.$$

2.2 A look at the integral (1.2).

If we consider, for $n \in \mathbb{N}$, the improper integral K_n, given by:

$$K_{n+1} = \int_0^\infty \frac{x}{(ax^2+2bx+c)^{n+1}}dx, \quad a>0. \tag{2.17}$$

Through an algebraic manipulation, K_{n+1} can be written as:

$$K_{n+1} = \frac{1}{2a}\int_0^\infty \frac{2ax+2b}{(ax^2+2bx+c)^{n+1}}dx - \frac{b}{a}I_{n+1},$$

the first integral on the right-hand side of the preceding equality is evaluated using the substitution $t = ax^2+2bx+c$. Thus we record the explicit evaluation of K_{n+1} i in the following result:

Proposition 2.2. *If K_{n+1} is given by (2.17), then*

$$K_{n+1} = \frac{1}{2anc^n} - \frac{b}{a}I_{n+1},$$

where I_{n+1} is the integral in Proposition 2.1.

Finally, we obtain an explicit expression for the integral (1.3) in the introduction.

2.3 A look at the integral (1.3)

For n a natural number, let S_{n+1} be the improper integral given by:

$$S_{n+1} = \int_0^\infty \frac{dx}{(ax^2+2bx+c)^{n+\frac{3}{2}}}, \tag{2.18}$$

in order to obtain a recurrence relation for the sequence $\{S_{n+1}\}$, let's consider the function $Q(x)$ defined by:

$$Q(x) = (2ax+2b)(ax^2+2bx+c)^{-n-1/2}, \tag{2.19}$$

differentiating (2.19) with respect to x:

$$\frac{dQ}{dx} = 2a(ax^2+2bx+c)^{-n-1/2} - \frac{2n+1}{2}(2ax+2b)^2(ax^2+2bx+c)^{-n-3/2},$$

taking into account that $(2ax+2b)^2$ can be written as:

$$(2ax+2b)^2 = 4a^2x^2+8abx+4b^2 = 4a(ax^2+2bx+c)+4(b^2-ac),$$

the equality (2.3) can be expressed as:

$$\frac{dQ}{dx} = \frac{-4an}{(ax^2+2bx+c)^{n+1/2}} - \frac{2(2n+1)(b^2-ac)}{(ax^2+2bx+c)^{n+3/2}}, \tag{2.20}$$

by integrating with respect to x on both sides of (2.20) and taking into account (2.19), we obtain:

$$\frac{b}{c^{n+1/2}} = 2an\int_0^\infty \frac{dx}{(ax^2+2bx+c)^{n+1/2}} + (2n+1)(b^2-ac)\int_0^\infty \frac{dx}{(ax^2+2bx+c)^{n+3/2}},$$

since S_{n+1} is defined by (2.18), the above equality can be written as:

$$S_{n+1} = \frac{b}{(2n+1)(b^2-ac)c^{n+1/2}} - \frac{2an}{(2n+1)(b^2-ac)}S_n, \tag{2.21}$$

by identifying the sequences $\{c_n\}$ and $\{d_n\}$ through:

$$c_n = \frac{2an}{(2n+1)(b^2-ac)}; \quad d_n = \frac{b}{(2n+1)(b^2-ac)c^{n+1/2}},$$

applying the recurrence formula (see [5]), we obtain:

$$S_{n+1} = \frac{(-2a)^2 n!}{(b^2-ac)^n(2n+1)!!}\left[S_1 + \sum_{k=1}^{n} \frac{b}{\sqrt{c}}\frac{(b^2-ac)^{k-1}(2k-1)!!}{(-2ac)^k k!}\right], \tag{2.22}$$

where

$$S_1 = \int_0^\infty \frac{dx}{(ax^2+2bx+c)^{3/2}},$$

taking into account that

$$S_1 = \frac{ax+b}{(ac-b^2)\sqrt{ax^2+2bx+c}}\Big|_0^\infty = \frac{1}{\sqrt{c}(\sqrt{ac}+b)}, \text{ with b} > -\sqrt{\text{ac}} \quad (2.23)$$

From the previous discussion, (2.22) and (2.23) allow us to establish the following assertion:

Proposition 2.3. *For $n \in \mathbb{N}$, $a > 0, c > 0, b > -\sqrt{ac}$*

$$\int_0^\infty \frac{dx}{(ax^2+2bx+c)^{n+3/2}} = \left(\frac{2ac}{ac-b^2}\right)^n \frac{n!}{c^{n+1/2}(\sqrt{ac}+b)(2n+1)!!} \left[1 + \frac{b}{(b-\sqrt{ac})}\sum_{k=1}^{n}\left(\frac{ac-b^2}{2ac}\right)^k \frac{(2k-1)!!}{k!}\right]. \quad (2.24)$$

Again, as in the case of *Propositions 2.1 and 2.2*, we obtain expressions for evaluating this type of integrals in a simpler way than those appearing in Table [4].

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