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3-Distance Domination Numbers of Some Graphs

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Abstract

Let G be a graph with vertex-set and edge-set $V(G)$ and $E(G)$, respectively. Then a subset Q of $V(\mathrm{G})$ is said to be a 3-distance dominating if for every $u \in V(G)\backslash Q$, there exists $v \in Q$ such that $d_G(u,v) = 3$. The minimum cardinality among all 3-distance dominating sets of G, denoted by $\gamma^3(G)$, is called the 3-distance domination number of G. In this paper, we obtain some bounds and exact values of the parameter for some graphs via characterization. Moreover, we characterize the equality of this parameter with respect to the domination and hop domination parameters of a graph, respectively.

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1 Introduction

Domination in graphs is one of the fastest growing areas in Graph Theory. Many authors contribute several interesting domination parameters to nurture the growth of this research area. Let G be a graph. A subset D of $V(G)$ is called a dominating of G if for every $v \in V(G)\backslash D$, there exists $u \in D$ such that $uv \in E(G)$; that is, a set D is called a dominating set of G if $N_G[D] = V(G)$. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets of G.

One of the several considered variations of the standard domination concept is hop domination introduced by Natarajan et al. [19]. A subset S of a vertices of a graph G is called a hop dominating if for every $a \in V(G) \setminus S$, there exists $b \in S$ such that $d_G(a, b) = 2$. The minimum cardinality among all hop dominating sets of G, denoted by $\gamma_h(G)$, is called the hop domination number of G. Researchers in the field had further investigated this concept, introduced new variants, and obtained some significant results that contributed a lot to the hop domination theory. Some studies related to hop domination and its variations can be found in [1, 2, 3, 4, 5, 6, 7, 8].

In this paper, 3-distance domination in a graph is introduced and investigated. Let G be a graph. Then a subset Q of $V(G)$ is said to be a 3-distance dominating if for every $u \in V(G)\backslash Q$, there exists $v \in Q$ such that $d_G(u, v) = 3$. The minimum cardinality among all 3-distance dominating sets of G, denoted by $\gamma^3(G)$, is called the 3-distance domination number of G. A vertex v in G is a *3-distance neighbor* of vertex u in G if $d_G(u, v) = 3$. The set $N^3_G(u) = \{v \in V(G) : d_G(v, u) = 3\}$ is called the *3-distance open neighborhood* of u. The *3-distance closed neighborhood* of u in G is given by $N^3_G[u] = N^3_G(u) \cup \{u\}$. The *3-distance open neighborhood* of $X \subseteq V(G)$ is the set $N^3_G(X) = \bigcup_{u\in X} N^3_G(u)$. The*3-distance closed neighborhood* of X in G is the set $N^3_G[X] = N^3_G(X) \cup X$. That is, Q is a 3-distance dominating if $N^3_G[Q] = V(G)$. We believe this study and its results would give additional insight and would serve as a reference for future researchers who might study some variations of domination.

2 Main results

Theorem 2.1. *Let G be a graph. Then $1 \leq \gamma^3(G) \leq |V(G)|$. Moreover, each of the following holds:*

i. $\gamma^3(G) = 1$ *if and only if G is trivial.*

ii. $\gamma^3(G) = |V(G)|$ *if and only if $diam(H) \leq 2$ for each component H of G.*

Proof. Let G be a graph. Since $\varnothing \subseteq V(G)$ is not 3-distance dominating, $\gamma^3(G) \geq 1$. Since any 3-distance dominating set Q is always a subset of $V(G)$, it follows that $\gamma^3(G) \leq |V(G)|$. Consequently, $1 \leq \gamma^3(G) \leq |V(G)|$.

(i) Suppose that $\gamma^3(G) = 1$. Then $\{v_i\} \subseteq V(G) = \{v_1, \cdots, v_n\}$ is minimum 3-distance dominating set of G for some $i \in \{1, 2, \cdots, n\}$. This means that $N^3_G[v_i] = V(G)$. If G is non-trivial, then $|V(G)| \geq 2$. Assume that G is disconnected and that $G_1, \cdots, G_k, k \geq 2$ are components of G. Assume that $v_i \in V(G_i)$ for some i. Now $V(G_j) \nsubseteq N^3_G[v_i]$ $\forall i \neq j$, a contradiction. Suppose that G is connected. Let $a \in N_G(v_i)$. Then $a \notin N^3_G[v_i]$, showing that $N^3_G[v_i] \neq V(G)$, a contradiction. Therefore, G is trivial.

Conversely, suppose that G is trivial. Then $\gamma^3(G) = 1$.

(ii) Suppose that $\gamma^3(G) = |V(G)|$. Then $V(G)$ is the minimum 3-distance dominating set of G. This means that $N^3_G[x] = \{x\}$ $\forall$ $x \in V(G)$. Suppose there is a component H of G with $diam(H) \geq 3$. Let $u, v \in V(H)$ such that $d_G(u, v) = 3$. Then $V(G)\backslash\{u\}$ is a 3-distance dominating set of G. Thus $\gamma^3(G) \leq |V(G)| - 1$, a contradiction. Therefore, $diam(H) \leq 2$ for every component H of G.

Conversely, suppose that $diam(H) \leq 2$ for each component H of G. Then $N^3_H[w] = \{w\}$ for each $w \in V(H)$. It follows that $N^3_G[a] = \{a\}$ for each $a \in V(G)$. Thus $V(G)$ is the only 3-distance dominating set of G. Consequently, $\gamma^3(G) = |V(G)|$. □

Corollary 2.2. *Let n and m be positive integers. Then each of the following holds:*

i. $\gamma^3(S_n) = n + 1$ $\forall$ $n \geq 1$.

ii. $\gamma^3(W_n) = n + 1$ $\forall$ $n \geq 3$.

iii. $\gamma^3(F_n) = n + 1$ $\forall$ $n \geq 2$.

iv. $\gamma^3(K_{m,n}) = m + n$ $\forall$ $n, m \geq 1$.

v. $\gamma^3(G + H) = |V(G + H)|$ *for any graphs G and H.*

Theorem 2.3. *Let G be a graph. If $\gamma_h(G) = |V(G)|$, then $\gamma^3(G) = |V(G)|$. However, the converse is not always true.*

Proof. Suppose that $\gamma_h(G) = |V(G)|$. Then $V(G)$ is the minimum hop dominating set of G. Assume first that G is connected. If G is non-complete, then there exist $x, y \in V(G)$ such that $d_G(x, y) = 2$. Thus, $V(G)\backslash\{y\}$ is a hop dominating set of G and so $\gamma_h(G) \leq |V(G)| - 1$, a contradiction. Therefore, G must be complete and so by Theorem 2.1, $\gamma^3(G) = |V(G)|$. Now, suppose that G is disconnected. Let $H_1, \ldots H_r$, $r \geq 2$ be components of G. Assume that H_i is non-complete for some $i \in \{1, \ldots, r\}$. Then $d_H(u, v) = 2$ for some $u, v \in V(H_i)$. Let $S = V(G)\backslash\{v\}$. Then S is a hop dominating set of G. Thus $\gamma_h(G) \leq |V(G)| - 1$, a contradiction. Hence every component H of G is complete and so $diam(H) \leq 1$. By Theorem 2.1, $\gamma^3(G) = |V(G)|$.

To see that the converse is not true, consider the graph G in Figure 1. Let $Q_1 = \{a, b, c, d, e\}$. Then by Theorem 3 , $\gamma^3(G) = 5$. Now, let $Q_2 = \{b, c, d, e\}$. Then $N^2_G[Q_2] = V(G)$. Thus Q_2 is a hop dominating set of G. Hence, $\gamma_h(G) \leq 4 < 5 = |V(G)| = \gamma^3(G)$. ☐

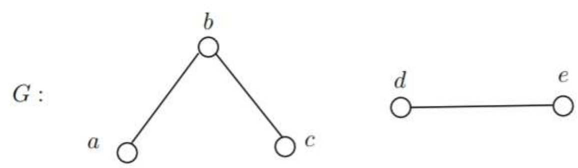

Figure 1: Graph G with $\gamma_h(G) \leq 4 < 5 = |V(G)| = \gamma^3(G)$

Theorem 2.4. *Let G be graph. Then $\gamma^3(G) = \gamma_h(G) = |V(G)|$ if and only if every component of G is complete.*

Proof. Suppose that $\gamma^3(G) = |V(G)| = \gamma_h(G)$. Then $diam(H) \leq 2$ for each component H of G by Theorem 2.1. Suppose that $diam(K) = 2$ for some component K of G. Let $a, b \in V(K) \subseteq V(G)$ such that $d_k(a, b) = 2 = d_G(a, b)$. Then $V(G)\backslash\{b\}$ is a hop dominating set of G. Thus $\gamma_h(G) \leq |V(G)| - 1$, a contradiction. Therefore, $diam(K) \leq 1$ for each component K of G. It follows that K is complete.

Conversely, suppose that each component H of G is complete. Let $x \in V(H)$. Then $N^2_H[x] = \{x\}$. Since x is arbitrary, it follows that $V(H)$ is the only hop dominating set of H. Thus $\gamma_h(H) = |V(H)|$. Since H is arbitrary, we have $\gamma_h(G) = |V(G)|$. Now, since every component H of G is complete, it follows that $diam(H) \leq 1$. Therefore, $\gamma^3(G) = |V(G)|$ by Theorem 2.1. ☐

Theorem 2.5. *Let G be a graph. Then $\gamma^3(G) = |V(G)| = \gamma(G)$ if and only if every component K of G is trivial.*

Proof. Suppose that $\gamma^3(G) = |V(G)| = \gamma(G)$. Since $\gamma^3(G) = |V(G)|$, $diam(K) \leq 2$ for each component K of G by Theorem 2.1. Assume that $diam(K) = 2$. Let $a, b \in V(K)$ such that $d_K(a, b) = 2$. Then $u \in N_K(a) \cap N_K(y)$ for some $u \in V(K)$. Let $Q = V(G)\backslash\{u\}$. Then Q is a dominating set of G. Thus $\gamma(G) \leq |V(G)| - 1$, a contradiction. Suppose that $diam(K) = 1$. Then there exist $x, y \in V(K)$ such that $d_K(x.y) = 1$. Let $P = V(G)\backslash\{x\}$. Then P is a dominating set of G. Hence $\gamma(G) \leq |V(G)| - 1$, a contradiction. Therefore, $diam(K) = 0$ for each component K of G. Consequently, K is trivial.

Conversely, suppose that every component K of G is trivial. Then $diam(K) = 0$. Thus, by Theorem 2.1, $\gamma^3(G) = |V(G)|$. Clearly, $\gamma(G) = |V(G)|$. That is, $\gamma^3(G) = |V(G)| = \gamma(G)$. $\square$

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