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Vertex Cover Zero Forcing Sets in Graphs

Javier A. Hassan, Ladznar S. Laja

Mathematics and Sciences Department
College of Arts and Sciences
MSU-Tawi-Tawi College of Technology and Oceanography
Bongao, Tawi-Tawi, Philippines

email: javierhassan@msutawi-tawi.edu.ph
ladznarlaja@msutawi-tawi.edu.ph

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Abstract

In this paper, we initiate the study of vertex cover zero forcing sets in graphs. We obtain some exact values of this parameter on some special graphs. Moreover, we characterize the vertex cover zero forcing sets in some special graphs and the join of two graphs. Furthermore, we obtain the formula for solving the vertex cover zero forcing number of each of these graphs.

1 Introduction

The concept of zero forcing has been explored over the past few years because of its application to minimum rank problems in linear algebra [2]. The color change rule states that a blue vertex adjacent to a single white neighbor can force its neighbor to blue. Formally, if u is a blue vertex and w is the only white vertex in $N(u)$, then $u \longrightarrow w$ will be used to denote that u forces w blue. A *zero forcing set* for a graph G is a subset Z of $V(G)$ such that if initially the vertices in Z are colored blue and the remaining vertices are colored white, the entire graph G may be colored blue by repeatedly applying

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the color-change rule. Moreover, the *zero forcing number*, $Z(G)$, of a graph G is the minimum cardinality of a set of blue vertices(whereas vertices in $V(G) \setminus S$ are colored white) such that $V(G)$ is turned blue after finitely many applications of "the color change rule": a white vertex is converted to a blue vertices if it is the only white neighbor of a blue vertex. Some studies on zero forcing sets and its variants can be found in [1, 2, 4, 5, 6, 7].

OLn the other hand, a vertex cover is a subset Q of a vertex-set $V(G)$ of G in which every edge in G is incident with some vertex in Q. The minimum cardinality of a vertex cover set of G is called a vertex cover number of G.

In this paper, we introduce and investigate a vertex cover zero forcing set in a graph. We believe this study and its results would serves as reference and would give additional insights to the future researchers who will study some variants of zero forcing and vertex cover of a graph.

2 Main results

We begin this section by introducing the concept of vertex cover zero forcing of a graph. The definition is as follows:

Definition 2.1. *Let G be a graphs. Then $X \subseteq V(G)$ is called a vertex cover zero forcing set in G if X is both a vertex cover and zero forcing set of G. The vertex cover zero forcing number of G, denoted by $Z_{vc}(G)$, is the minimum cardinality among all vertex cover zero forcing sets in G.*

Example: Consider the graph G in Figure 1. Let $X = \{v_2, v_4, v_6, v_8\}$. Then X is a vertex cover of G. Now, observe that v_7, v_5, v_3, v_1 are forcing by v_8, v_6, v_4, v_2, respectively. Hence X is a zero forcing set of G. Therefore, X is a vertex cover zero forcing set of G. Note that X is a minimum vertex cover zero forcing set of G and so $Z_{vc}(G) = 4$.

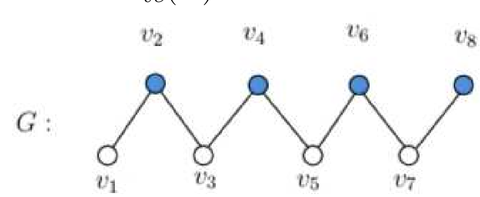

Figure 1: Graph G with $Z_{vc}(G) = 4$

Remark 2.2. *Let G be a graph. Then $1 \leq Z_{vc}(G) \leq |V(G)|$.*

Theorem 2.3. *Let G be a graph. Then $Z_{vc}(G) = |V(G)|$ if and only if $G = \overline{K}_n$.*

Proof. Suppose that $Z_{vc}(G) = |V(G)|$. Then $V(G)$ is the only vertex cover zero forcing set of G. Assume first that G is connected. Suppose that $G \neq K_1$. Then $V(G)\setminus\{a\}$ is a vertex cover zero forcing set of G for some $a \in V(G)$. Thus $Z_{vc}(G) \leq |V(G)| - 1$, a contradiction. Therefore, $G = K_1$. Next, assume that G is disconnected. If G has non-trivial component, say K, then $V(G) \setminus \{x\}$ is a vertex cover zero forcing set of G for some $x \in V(K)$. Thus $Z_{vc}(G) \leq |V(G) - 1|$, a contradiction. Hence every component of G is trivial and so $G = \overline{K}_n$.

The converse is clear. □

Theorem 2.4. *Let K_n be a complete graph of order $n \geq 2$. Then $X \subseteq V(K_n)$ is a vertex cover zero forcing if and only if $|X| \geq n - 1$.*

Proof. Suppose that X is a cover vertex zero forcing set of K_n. Then X is a zero forcing set of K_n. Suppose on the contrary that $|X| \leq n - 2$. Then there are at least two vertices in $V(K_n)$. which are not in X. Let these two vertices be u and v. Then any of the vertices in $V(K_n) \setminus \{u, v\}$ cannot force u and v, a contradiction. Therefore, $|X| \geq n - 1$.

Conversely, suppose that $|X| \geq n-1$. If $|X| = n$, then X is a cover vertex zero forcing set of G. Hence we are done. Next, assume that $|X| = n - 1$. Then $X = V(K_n) \setminus \{u\}$ for some $u \in V(K_n)$. Clearly, X is a zero forcing set of K_n. Now, since K_n is complete, all edges incidents to u are alsp incidents to all vertices in $V(K_n) \setminus \{u\}$. This means that X is a vertex cover of K_n. Consequently, X is cover vertex zero forcing set of K_n. □

Corollary 2.5. *Let n be any positive integer. Then* $Z_{vc}(K_n) = \begin{cases} 1, & n = 1 \\ n-1, & n \geq 2. \end{cases}$

Theorem 2.6. *Let n be a positive integer. Then* $Z_{vc}(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & n \text{ is odd} \\ \frac{n}{2}, & n \text{ is even.} \end{cases}$

Proof. Clearly, $Z_{vc}(P_1) = 1$ and $Z_{vc}(P_3) = 2$. Suppose that $n \geq 5$ is odd. Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ and let $Q = \{v_2, v_4, \cdots, v_{n-1}, v_n\}$. Then Q is a minimum vertex cover of P_n. Clearly, Q is also a zero forcing set of P_n. Thus Q is a minimum vertex cover zero forcing set of P_n. Therefore, $Z_{vc}(P_n) = \lceil \frac{n}{2} \rceil$ for all odd integers $n \geq 5$.

Next, clearly $Z_{vc}(P_2) = 1$ and $Z_{vc}(P_4) = 2$. Suppose that $n \geq 6$ is even. Let $N = \{v_2, v_4, \cdots, v_n\}$. Then N is a minimum vertex cover of P_n. Now, vertices $v_{n-1}, v_{n-3} \cdots, v_3$ and v_1 are forced by the vertices $v_n, v_{n-2} \cdots, v_{n-4}$ and v_2, respectively. It follows that N is a zero forcing set of P_n. Thus N

is a minimum vertex cover zero forcing set of P_n. Hence $Z_{vc}(P_n) = \frac{n}{2}$ for all even integers $n \geq 6$. □

Theorem 2.7. *Let n be a positive integer. Then* $Z_{vc}(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & n \text{ is odd} \\ \frac{n}{2} + 1, & n \text{ is even.} \end{cases}$

Proof. Clearly, $Z_{vc}(C_3) = 2$. Suppose that $n \geq 5$ is odd. Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ and let $S = \{v_1, v_3, \cdots, v_n\}$. Then S is a minimum vertex cover of C_n. Notice that N vertices $v_2, v_4, \cdots, v_{n-3}$ and v_{n-1} are forced by the vertices $v_1, v_3, \cdots, v_{n-4}$ and v_{n-2}, respectively. It follows that N is a zero forcing set of C_n. Therefore, S is a minimum vertex cover zero forcing set of C_n; that is, $Z_{vc}(C_n) = \lceil \frac{n}{2} \rceil$ for all odd integers $n \geq 5$.

Next, clearly $Z_{vc}(C_4) = 3$. Suppose that $n \geq 6$ is even. Let $M = \{v_1, v_3, \cdots, v_{n-1}, v_n\}$. Then M is a minimum vertex cover zero forcing set of C_n. Hence $Z_{vc}(C_n) = \frac{n}{2} + 1$ for all even integers $n \geq 6$. □

Theorem 2.8. *Let S and T be graphs. Then $X \subseteq V(S+T)$ is a vertex cover zero forcing set of $S+T$ if and only if $X = X_S \cup X_T$ and satisfies one of the following conditions:*

i. $X_S = V(S)$ and X_T is a vertex cover zero forcing set of T.

ii. $X_T = V(T)$ and X_S is a vertex cover zero forcing set of S.

Proof. Suppose that X is a vertex cover zero forcing set of $S+T$. If $X_S = V(S)$ and $X_T = V(T)$, then we are done. Suppose that $X_S \neq V(S)$. Then there exists $x \in V(S) \setminus X_S$ such that $x \notin X$. If $X_T \neq V(T)$. Then there exists $y \in V(T) \setminus X_T$ such that $y \notin X$. Note that $xy \in E(S+T)$ but $x \notin X$ and $y \notin X$, a contradiction to the fact that X is a vertex cover of $S+T$. Thus$X_T = V(T)$. Now, since X is a vertex cover zero forcing set of $S+T$, X_S is a vertex cover zero forcing set of S. Therefore,(ii) holds. Similarly, (i) holds.

The converse is clear. □

Corollary 2.9. *Let S and T be graphs. Then*

$$Z_{vc}(S+T) = min\{|V(S)| + Z_{vc}(T), |V(T) + Z_{vc}(S)\}.$$

Moreover, each of the following follows:

i. $Z_{vc}(K_n + P_n) = Z_{vc}(P_n + P_n) = \begin{cases} n + \lceil \frac{n}{2} \rceil, & if\ n\ is\ odd \\ n + \frac{n}{2}, & if\ n\ is\ even. \end{cases}$

ii. $Z_{vc}(K_n + K_m) = \begin{cases} 2, \ if \ n,m = 1 \\ m, \ if \ n = 1 \ and \ m \geq 2 \\ n, \ if \ n \geq 2 \ and \ m = 1 \\ n + m - 1, \ if \ n,m \geq 2. \end{cases}$

iii. $Z_{vc}(K_n + C_n) = Z_{vc}(C_n + C_n) = \begin{cases} n + \lceil \frac{n}{2} \rceil, \ if \ n \ is \ odd \\ \frac{3n}{2} + 1, \ if \ n \ is \ even. \end{cases}$

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