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2-Domination Zero Forcing in Graphs

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Abstract

Let G be a graph with vertex and edge-sets $V(G)$ and $E(G)$, respectively. Then $T \subseteq V(G)$ is a called 2-dominating zero forcing set of G if T is a zero forcing set of G and for every $u \in V(G) \setminus T$, u has at least 2-neighbors in T. The 2-domination zero forcing number of G, denoted by $Z_2(G)$, is the minimum cardinality of a 2-dominating zero forcing set of G. In this paper, we investigate this concept on some special graphs, join and corona of two graphs. Moreover, we formulate some formulas for calculating the parameter of these graphs. Furthermore, we characterize this type of set in the join of two graphs and finally find a simplified formula for solving this parameter on this graph.

1 Introduction

A dominating set in a graph is a subset of vertices such that every vertex in the graph is either in the subset or adjacent to at least one vertex in the subset. A dominating set is said to be minimal if no proper subset is also a dominating set. The size of a minimal dominating set is called the domination

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number of the graph. The domination number of a graph is an important parameter that has applications in various fields such as computer science, social network analysis, and operations research. For example, in computer networks, a dominating set can be used to monitor the network traffic, and in social networks, a dominating set can represent influential individuals who can spread information quickly. In addition, several variations of the domination problem have been studied (see [3, 4, 5, 6, 7, 8, 9, 10]).

Zero forcing sets in graphs comprise an interesting concept in graph theory. A zero forcing set in a graph is a subset of vertices with a particular dynamic propagation property. The zero forcing process starts with a set of initially colored vertices, typically with one color representing "active" and another "inactive" or "unassigned." Then, using certain propagation rules, the active vertices force neighboring inactive vertices to become active. The goal is to determine the minimum size of a zero forcing set required to force all vertices in the graph to become active. Zero forcing sets have applications in control theory, network coding, and determining structural properties of graphs. Some studies on zero forcing sets in graphs can be found in [1, 2].

In this paper, a new parameter called 2-domination zero forcing in a graph is introduced and investigated. The formulation of this parameter is motivated by the notions of 2-domination and zero forcing sets in a graph. We believe, this parameter and its results would give additional insights to future researchers who might study concepts related to 2-domination and zero forcing sets in graphs.

2 Main results

We begin this section by formally introducing the 2-domination zero forcing in a graph. The definition is given as follows:

Definition 2.1. Let G be a graph with vertex and edge-sets $V(G)$ and $E(G)$, respectively. Then $T \subseteq V(G)$ is a called 2-dominating zero forcing set of G if T is a zero forcing set of G and for every $u \in V(G) \setminus T$, u has at least 2-neighbors in T. The 2-domination zero forcing number of G, denoted by $Z_2(G)$, is the minimum cardinality of a 2-dominating zero forcing set of G.

To elaborate further on the aforementioned concept, take a look at the given example below.

Example 2.2. Consider the $P_7 = G$ below.

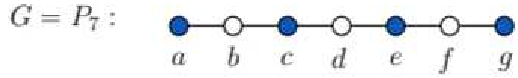

Figure 1: Graph G with $Z_2(G) = 4$

Let $T = \{a, c, e, g\}$. Then vertices b, d and f are forced by vertices a, c and e, respectively. It follows that T is a zero forcing set of G. Notice that vertices b, d and f have 2 neighbors in T. Therefore, T is a 2-dominating zero forcing set of G. Moreover, it can be verified that $Z_2(G) = 4$.

Remark 2.3. *Let G be a graph. Then*

(i) $Z(G) \leq Z_2(G)$;

(ii) $\gamma(G) \leq Z_2$;

(iii) $1 \leq Z_2(G) \leq |V(G)|$;

(iv) if $S \subseteq V(G)$ is a 2-dominating and minimum zero forcing set of G, then $Z_2(G) = |S|$; and

(v) if $Q \subseteq V(G)$ is a zero forcing and minimum 2-dominating set of G, then $Z_2(G) = |Q|$.

Theorem 2.4. *Let n be a positive integer. Then*

$$Z_2(P_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + 1 \ , & \textit{if } n \textit{ is odd} \\ \frac{n}{2} + 1 \ , & \textit{if } n \textit{ is even} \end{cases}$$

Proof. Let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$. Clearly, $Z_2(P_1) = 1$ and $Z_2(P_3) = 2$. Suppose that $n \geq 5$ and odd. Consider $N = \{v_1, v_3, \ldots, v_n\}$. Then N is a minimum 2-dominating set of P_n. Note that vertices $v_2, v_4 \ldots, v_{n-3}$ and v_{n-1} are forced by vertices $v_1, v_3 \ldots, v_{n-4}$ and v_{n-2}, respectively. It follows that N is a zero forcing set of P_n. Thus N is a minimum 2-dominating zero forcing set of P_n. Since n is odd, it follows that $|N| = \left\lfloor \frac{n}{2} \right\rfloor + 1 = Z_2(P_n)$ for all $n \geq 5$ and odd.

Next, clearly $Z_2(P_2) = 2$ and $Z_2(P_4) = 3$. Suppose that $n \geq 6$ and even. Let $Q = \{v_1, v_3, \ldots, v_{n-1}, v_n\}$. Then Q is a minimum 2-dominating set of P_n. Now, observe that vertices $v_2, v_4, \ldots, v_{n-4}$ and v_{n-2} are forced by vertices $v_1, v_3, \ldots, v_{n-5}$ and v_{n-3}, respectively. Thus Q is a zero forcing set of P_n and so Q is a minimum 2-dominating zero forcing set of P_n. Since n is even, it follows that $|Q| = \frac{n}{2} + 1 = Z_2(P_n)$ for all $n \geq 6$ and even. $\square$

Theorem 2.5. *Let n be a positive integer. Then*

$$Z_2(C_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + 1, \textit{if } n \textit{ is odd} \\ \frac{n}{2} + 1, \textit{if } n \textit{ is even} \end{cases}$$

Proof. Let $V(C_n) = \{a_1, a_2, \ldots, a_n\}$. Clearly, $Z_2(C_3) = 2$. Suppose that $n \geq 5$ and odd. Consider $S = \{a_1, a_3, \ldots, a_n\}$. Then S is a minimum 2-dominating set of C_n. Now, vertices $a_2, a_4, \ldots, a_{n-3}$ and a_{n-1} are forced by vertices $a_1, a_3, \ldots, a_{n-4}$ and a_{n-2}, respectively. It follows that S is a zero forcing set of C_n. Thus S is a minimum 2-dominating zero forcing set of C_n. Since n is odd, $|S| = \left\lfloor \frac{n}{2} \right\rfloor + 1 = Z_2(C_n)$ for all $n \geq 5$ and odd.

Next, clearly $Z_2(C_4) = 3$. Suppose that $n \geq 6$ and even. Let $T = \{a_1, a_3, \ldots, a_{n-1}, a_n\}$. Then T is a minimum 2-dominating zero forcing set of C_n. Since n is even, $|T| = \frac{n}{2} + 1 = Z_2(C_n)$ for all $n \geq 6$ and even. □

Theorem 2.6. *Let n be a positive integer. Then*

$$Z_2(K_n) = \begin{cases} n & , \ n = 1, 2 \\ n - 1 & , \ n \geq 3 \end{cases}$$

Proof. Clearly, $Z_2(K_1) = 1$. For $n = 2$, let $V(K_2) = \{u_1, u_2\}$. Let S be a 2-dominating zero forcing set of K_2. Suppose that $Z_2(K_2) = 1$. Then either $S = \{u_1\}$ or $S = \{u_2\}$. If $S = \{u_1\}$, then $u_2 \in V(K_2) \setminus S$ has only one neighbor u_1 in S. However, this is a contradiction to the fact that S is a 2-dominating set of K_2. Similarly when $S = \{u_2\}$. Therefore, $Z_2(K_2) = 2$.

Next, suppose that $n \geq 3$. Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and consider $N = \{u_1, u_2, \ldots u_{n-1}\}$. Then N is a minimum zero forcing set of K_n. Clearly, N is a 2-dominating set of K_n. Therefore, N is a minimum 2-dominating zero forcing set of K_n and so $Z_2(K_n) = n - 1$ for all $n \geq 3$. □

Theorem 2.7. *Let G and H be non-complete graphs with order of at least three. Then $Q \subseteq V(G + H)$ is a 2-dominating zero forcing set of $G + H$ if and only if $Q = Q_G \cup Q_H$ and satisfies one of the following conditions:*

(i) $Q_G = V(G)$ *and* Q_H *is a zero forcing of* H.

(ii) $Q_H = V(H)$ *and* Q_G *is a zero forcing set of* G.

(iii) $Q_G = V(G) \setminus \{x\}$ *and* Q_H *is a zero forcing set of* H *with* $N_H[y] \cap (V(H) \setminus Q_H) = \emptyset$ *for some* $y \in Q_H$ *and* $x \in V(G)$.

(iv) $Q_H = V(H) \setminus \{u\}$ *and* Q_G *is a zero forcing set of* G *with* $N_G[v] \cap (V(G) \setminus Q_G) = \emptyset$ *for some* $v \in Q_G$ *and* $u \in V(G)$.

Proof. Let Q be a 2-dominating zero forcing set of $G+H$. If $Q_G = V(G)$ and $Q_H = V(H)$, then $Q = V(G+H)$, and we are done. Suppose that $Q_G = V(G)$. Since $Q = Q_G \cup Q_H = V(G) \cup Q_H$ is a 2-dominating zero forcing set of $G+H$, it follows that Q_H is a zero forcing set of H. Hence,(i) holds. Similarly, (ii) holds. Now, suppose that $Q_G \neq V(G)$ and $Q_H \neq V(H)$. Since Q is zero forcing, either $|Q_G| = |V(G)| - 1$ or $|Q_H| = |V(H)| - 1$. If $|Q_G| = |V(G)| - 1$ and $|Q_H| = |V(H)| - 1$, then either (iii) or (iv) holds since G and H are non-complete graphs. So, assume that $|Q_G| = |V(G)| - 1$. and $|Q_H| \leq |V(H)| - 2$. Suppose that $w \in Q_H$, where $N_H[w] \cap (V(H) \setminus Q_H) \neq \emptyset$. Then any $z \in Q$ cannot force either x or any element in $V(H) \setminus Q_H$, which is a contradiction. Now, since Q is a zero forcing set of $G+H$, Q_H must be a zero forcing set in H. That is, (iii) holds. Similarly, (iv) follows.

Conversely, the assertion follows when (i) or (ii) holds. Next, suppose that (iii) holds. Since $|V(G)| \geq 3$, it follows that Q is a 2-dominating set of $G+H$. Moreover, y forces vertex x in G. Since Q_H is a zero forcing set in H, it follows that $Q = Q_G \cup Q_H$ is a zero forcing set of $G+H$. Therefore, Q is a 2-dominating zero forcing set of $G+H$. Similarly, the assertion follows, when (iv) is true. □

Corollary 2.8. *Let* G *and* H *be non-complete graphs with order of at least three. Then*

$$Z_2(G+H) = min\{|V(G)| + Z(H), |V(H)| + Z(G)\}.$$

Theorem 2.9. *Let* G *and* H *be graphs. Then* $N \subseteq V(G \circ H)$ *is a 2-dominating zero forcing set of* $G \circ H$ *if* $N = V(G) \cup (\bigcup_{v \in V(G)} A_v)$, *where* A_v *is a 2-dominating zero forcing set in* H^v *for each* $v \in V(G)$. *Moreover,* $Z_2(G \circ H) \leq |V(G)| + Z_2(H)|V(G)|$.

Proof. Suppose that $N = V(G) \cup (\bigcup_{v \in V(G)} A_v)$, where A_v is a 2-dominating zero forcing set in H^v for each $v \in V(G)$. Let $x \in V(G \circ H) \setminus N$. Then $x \in V(H^u)$ for some $u \in V(G)$. Since A_u is 2-dominating, there exist $y, w \in A_u$ such that $d_{H^u}(x,y) = 1$ and $d_{H^u}(x,w) = 1$. Thus N is a 2-dominating set of $G \circ H$. Since A_v is a zero forcing set of H^v for each $v \in V(G)$, it follows that N is a zero forcing set of $G \circ H$. Consequently, N is a 2-dominating zero forcing set of $G \circ H$. Now, since $Z_2(G \circ H)$ is the minimum cardinality of a 2-dominating zero forcing set in $G \circ H$, it follows that $Z_2(G \circ H) \leq |V(G)| + Z_2(H)|V(G)|$. □

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