International Journal of Mathematics and
Computer Science, **19**(2024), no. 4, 1105–1110

Certified Hop Domination in Graphs

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(Received March 2, 2024, Accepted April 5, 2024,
Published June 1, 2024)

Abstract

Let G be a simple and undirected graph. A subset C of a vertex-set $V(G)$ of G is called a certified hop dominating if C is hop dominating and for every $u \in C$, u has either zero or at least two neighbors in $V(G) \setminus C$. The minimum cardinality of a certified hop dominating set of G, denoted by $\gamma_{cerh}(G)$, is called the certified hop domination number of G. In this paper, we introduce this parameter and we investigate this on some special graphs and the join of two graphs. Its relationships with hop domination, certified domination, and certified hop independence parameters of a graph are presented. Moreover, bounds and some formulas of this parameter are derived via characterizations.

Key words and phrases: Certified hop dominating set, certified hop domination number
AMS (MOS) Subject Classifications: 05C69.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

1 Introduction

The concept of hop domination in graphs is a relatively recent development in the field of graph theory emerging as an extension of classical domination theory. This parameter was introduced by Natarajan et al. [11]. Hop domination has applications in various fields such as network design, communication networks, and facility location problems. It helps in understanding the efficiency and coverage of networks with limited resources.

Researchers continue to explore various aspects of hop domination including its computational complexity, structural properties, and applications in real-world networks. As graph theory and its applications continue to evolve, hop domination remains an active area of research contributing to our understanding of network dynamics and optimization.

Researchers in the field have investigated this concept and its variants further and they have obtained some significant results that contributed a lot to the hop domination theory (see [1-10]).

In this paper, we introduce and investigate a new variant of hop domination called certified hop domination in a graph. We believe that the results here would give additional insights to future researchers in the field and would help them for more research directions in the future.

2 Main results

We begin this section by introducing the concept of certified hop domination in a graph.

Definition 2.1. *Let G be a simple and undirected graph. A subset C of a vertex-set $V(G)$ of G is called a certified hop dominating if C is hop dominating and for every $u \in C$, u has either zero or at least two neighbors in $V(G) \setminus C$. The minimum cardinality of a certified hop dominating set of G, denoted by $\gamma_{cerh}(G)$, is called the certified hop domination number in G.*

Remark 2.2. *Let G be a graph. Then*

(i) a hop dominating set Q of G may not be a certified hop dominating; and

(ii) every certified hop dominating set Q of G is a hop dominating. That is, $\gamma_{cerh}(G) \geq \gamma_h(G)$.

Theorem 2.3. *Let P_n be a path graph of order $n \geq 1$. Then Q is a certified hop dominating set of P_n if and only if $Q = V(P_n)$.*

Proof. Let P_n be a path graph of order n. Let Q be a certified hop dominating set of P_n, where $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. Assume that $v_1 \in Q$. Since Q is a certified set in P_n, v_2 must be in Q. However, v_2 has only one neighbor v_3 in $V(P_n) \setminus Q$. So v_3 must also be included in Q. Continuing this process, all vertices of P_n are in Q. Hence $Q = V(P_n)$ and we are done. Suppose that $v_1 \notin Q$. Since Q is a hop dominating, v_3 must be in Q in order to hop dominates v_1. Now, suppose that $v_2 \in Q$. Then v_2 has only one neighbor v_1 in $V(P_n) \setminus Q$, a contradiction since $v_1 \notin Q$. Thus, $v_2 \notin Q$. Since Q is a hop dominating, v_4 must be in Q. Since $d_{P_n}(v_3, v_4) = 1$, it follows that v_3 and v_4 have only one neighbor v_2 and v_5 in $V(P_n) \setminus Q$, respectively. Hence v_2 and v_5 must be in Q. Similarly, v_2 and v_5 have only one neighbor v_1 and v_6 in $V(P_n) \setminus Q$, respectively. Continuing this process, all other vertices of P_n will be included as elements of the set Q. Consequently, $Q = V(P_n)$.

The converse is clear. □

Corollary 2.4. *Let n be a positive integer. Then $\gamma_{cerh}(P_n) = n$ for all $n \geq 1$.*

Theorem 2.5. *Let G be a graph with $|V(G)| \geq 1$. Then*

(i) $1 \leq \gamma_{cerh}(G) \leq |V(G)|$;

(ii) $\gamma_{cerh}(G) = 1$ if and only if $|V(G)| = 1$;

(iii) if $\gamma_{cerh}(G) = 2$, then $\gamma_h(G) = 2$. However, the converse is not always true; and

(iv) if $\gamma_h(G) = |V(G)|$, then $\gamma_{cerh}(G) = |V(G)|$. However, the converse is not always true.

Proof. (i) Let G be a graph. Since $\varnothing$ is not a certified hop dominating set G, it follows that $\gamma_{cerh}(G) \geq 1$. Note that every certified hop dominating set Q of G is always a subset of a vertex-set $V(G)$ of G. Thus, $\gamma_{cerh}(G) \leq |V(G)|$. Consequently, $1 \leq \gamma_{cerh} \leq |V(G)|$.

(ii) Suppose that $\gamma_{cerh}(G) = 1$. Then $Q = \{x\}$ is a minimum certified hop dominating set of G. Thus $N_G^2[x] = V(G)$. Assume that $|V(G)| \geq 2$. Suppose that G is connected. Then there exists $a \in V(G)$ such that $d_G(a, x) = 1$. Thus $a \notin N_G^2[x]$, a contradiction. Next, suppose that G is disconnected. Let $G_1, \cdots, G_k, k \geq 2$ be components of G. Then

$$\gamma_{cerh}(G) = \gamma_{cerh}(G_1) + \ldots + \gamma_{cerh}(G_k) \geq 2,$$

a contradiction. Therefore, $|V(G)| = 1$.

Conversely, suppose that $|V(G)| = 1$. Then, by (i), $\gamma_{cerh}(G) = 1$.

(iii) Suppose that $\gamma_{cerh}(G) = 2$. Then, by (ii), G is non-trivial. Thus, $\gamma_h(G) \geq 2$. Since $\gamma_{cerh}(G) \geq \gamma_h(G)$, it follows that $\gamma_h(G) = 2$. To see that the converse is not true, consider P_3. Then $\gamma_h(P_3) = 2$. However, $\gamma_{cerh}(G) = 3$ by Corollary 2.4.

(iv) Suppose that $\gamma_h(G) = |V(G)|$. Since $\gamma_{cerh}(G) \geq \gamma_h(G)$ and $\gamma_{cerh}(G) \leq |V(G)|$, it follows that $\gamma_{cerh}(G) = |V(G)|$. To see that the converse is not true, consider P_4. Then $\gamma_{cerh}(P_4) = 4$ by Corollary 2.4 and $\gamma_h(P_4) = 2$. □

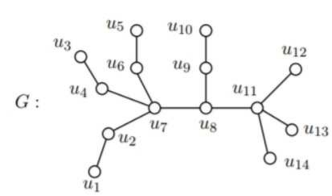
Theorem 2.6. *Let G be a graph. Then every maximum certified hop independent set Q of G is a certified hop dominating set of G. Moreover, $\gamma_{cerh}(G) \leq \alpha_{cerh}(G)$.*

Proof. Let G be a graph and let Q be a maximum certified hop independent set of G. Then $d_G(u, v) \neq 2$ for every $u, v \in Q$. Now, let $a \in V(G) \setminus Q$. Since Q is a maximum certifeid hop independent set G, it follows that $d_G(a, b) = 2$ for some $b \in Q$. This means that $a \in N_G^2(b)$; that is, $a \in N_G^2(Q)$. Hence $V(G) \setminus Q \subseteq N_G^2(Q)$. Therefore, $N_G^2[Q] = V(G)$ showing that Q is a hop dominating set of G. Since Q is a certified hop independent set of G, Q is a certifeid hop dominating set of G.

Now, let S be a maximum certified hop independent set of G. Then S is a certified hop dominating set of G. Thus $\alpha_{cerh}(G) = |S| \geq \gamma_{cerh}(G)$. □

Remark 2.7. *Let G be a graph. Then the certified hop domination and certified domination parameters of G are incomparable.*

Proof. To see this, consider the graph G below:



Let $S_1 = \{u_2, u_4, u_6, u_9, u_{11}\}$. Then S_1 is a minimum dominating set of G. Observe that $N_G(u_2) = \{u_1, u_7\}$, $N_G(v_4) = \{u_3, u_7\}$, $N_G(v_6) = \{u_5, u_7\}$, $N_G(v_9) = \{u_8, u_{10}\}$ and $N_G(v_{11}) = \{u_8, u_{12}, u_{13}, u_{14}\}$, where $u_1, u_3, u_5, u_7, u_8, u_{10}, u_{12}, u_{13}, u_{13} \in V(G) \setminus S_1$. It follows that S_1 is a minimum certified dominating set of G. Hence $\gamma_{cer}(G) = 5$. Now, let $S_2 = \{u_7, u_8\}$. Then $N_G^2[S_2] = V(G)$ showing that S_2 is a hop dominating set of G. Since Q is non-trivial, S_2 is a minimum hop dominating set of G. Notice that $N_G(u_7) = \{u_2, u_4, u_6, u_8\}$ and $N_G(u_8) = \{u_7, u_9, u_{11}\}$, where $u_2, u_4, u_6, u_9, u_{11} \in V(G) \setminus S_2$. Thus, S_2

is a minimum certified hop dominating set of G. Therefore, $\gamma_{cerh}(G) = 2$. Consequently, $\gamma_{cerh}(G) < \gamma_{cer}(G)$.

Next, consider a complete graph K_n, where $n \geq 3$. Then $\gamma_{cer}(K_n) = 1$. Since $\gamma_h(K_n) = n$ and $\gamma_{cerh}(G) \geq \gamma_h(G)$ for any graph G, it follows that $\gamma_{cerh}(K_n) = n \geq 3$. Hence $\gamma_{cerh}(K_n) > \gamma_{cer}(K_n)$. □

Theorem 2.8. *Let S and T be graphs. A set $Q \subseteq V(S+T)$ is called a certified hop dominating set of $S+T$ if and only if $Q = Q_S \cup Q_T$ and satisfies the following conditions:*

(i) Q_S and Q_T are pointwise non-dominating sets of S and T, respectively.

(ii) For all $a \in Q_S \subseteq V(S)$, $|N_{S+T}(a)\backslash Q| \geq 2$ or $|N_{S+T}(a)\backslash Q| = 0$.

(iii) For all $b \in Q_T \subseteq V(T)$, $|N_{S+T}(b)\backslash Q| \geq 2$ or $|N_{S+T}(b)\backslash Q| = 0$.

Proof. Suppose that Q is a certified hop dominating set of $S+T$. Let $x \in V(S+T)\backslash Q$. Then either $x \in V(S)\backslash Q_S$ or $x \in V(T)\backslash Q_T$. Assume that $x \in V(S) \setminus Q_S$. Since Q is a hop dominating, there exists $y \in Q_S \subseteq Q$ such that $d_{S+T}(x,y) = 2$, that is $d_S(x,y) \geq 2$. This means that $x \notin N_S(y)$. Since x is arbitrary, it follows that Q_S is a pointwise non-dominating set of S. Similarly, when $x \in V(T) \setminus Q_T$, then Q_T is a pointwise non-dominating set of T. Hence, (i) holds. Now, Let $a \in Q$. Then either $a \in Q_S$ or Q_T. Assume that $a \in Q_S$. Since Q is a certified hop dominating set $S+T$, $|N_{S+T}(a) \setminus Q| \geq 2$ or $|N_{S+T}(a) \setminus Q| = 0$. Thus (ii) holds. Next, suppose that $a \in Q_T$. Since Q is a certified hop dominating set of $S+T$, it follows that $|N_{S+T}(a) \setminus Q| \geq 2$ or $|N_{S+T}(a) \setminus Q| = 0$. Hence (iii) holds.

Conversely, suppose that (i), (ii) and (iii) hold. Let $u \in V(S+T) \setminus Q$. Then either $u \in V(S) \setminus Q_S$ or $u \in V(T) \setminus Q_T$. Suppose that $u \in V(S) \setminus Q_S$. Since Q_S is a pointwise non-dominating set of S, there exists $v \in Q_S \subseteq Q$ such that $d_S(u,v) \geq 2$. Thus, $d_{S+T}(u,v) = 2$. It follows that Q is a hop dominating set of $S+T$. Similarly, the assertion follows when $u \in V(T) \setminus Q_T$. Clearly, Q is a certified hop dominating set of $S+T$ by (ii) and (iii). □

Corollary 2.9. *Let S and T be graphs. Then $\gamma_{cerh}(S+T) \geq pnd(S) + pnd(T)$.*

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