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Almost (τ_1, τ_2)-continuous multifunctions

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Abstract

In this paper, we introduce the notion of almost (τ_1, τ_2)-continuous multifunctions. We also investigate several characterizations of almost (τ_1, τ_2)-continuous multifunctions.

1 Introduction

In 1968, M. K. Singal and A. R. Singal [11] introduced the notion of almost continuous functions as a generalization of continuity. Popa [10] defined almost quasi-continuous functions as a generalization of almost continuity [11] and quasi-continuity [7]. In 2001, Popa and Noiri [9] introduced the notion of

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almost m-continuous multifunctions and presented the relationships between m-continuity [8] and almost m-continuity. In 2020, Viriyapong and Boonpok [12] introduced and investigated the notion of almost $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Laprom et al. [6] introduced and studied the notion of almost $\beta(\tau_1, \tau_2)$-continuous multifunctions. Moreover, several characterizations of almost $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions and almost weakly (τ_1, τ_2)-continuous multifunctions were established in [4] and [3], respectively. In this paper, we introduce the notion of almost (τ_1, τ_2)-continuous multifunctions. In addition, we investigate some characterizations of almost (τ_1, τ_2)-continuous multifunctions.

2 Preliminaries

Throughout the paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [5] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [5] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [5] of A and is denoted by $\tau_1\tau_2$-Int(A).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3 Almost (τ_1, τ_2)-continuous multifunctions

We begin this section by introducing the notion of almost (τ_1, τ_2)-continuous multifunctions.

Definition 3.1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be almost (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open sets*

V_1, V_2 of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_1))$ and $\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_2)) \cap F(z) \neq \emptyset$ for every $z \in U$.

Theorem 3.2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is almost (τ_1, τ_2)-continuous;

(2)

$$F^+(V_1) \cap F^-(V_2) \subseteq \tau_1\tau_2\text{-}Int(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_1))) \cap F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_2))))$$

for every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y;

(3)

$$\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K_1))) \cup F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K_2)))) \subseteq F^-(K_1) \cup F^+(K_2)$$

for every $\sigma_1\sigma_2$-closed sets K_1, K_2 of Y;

(4)

$$\tau_1\tau_2\text{-}Cl[F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B_1)))) \cup F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B_2))))] \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B_1)) \cup F^+(\sigma_1\sigma_2\text{-}Cl(B_2))$$

for every subsets B_1, B_2 of Y;

(5)

$$F^+(\sigma_1\sigma_2\text{-}Int(B_1)) \cap F^-(\sigma_1\sigma_2\text{-}Int(B_2)) \subseteq \tau_1\tau_2\text{-}Int[F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B_1)))) \cap F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B_2))))]$$

for every subsets B_1, B_2 of Y.

Proof. (1) $\Rightarrow$ (2): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y such that

$$x \in F^+(V_1) \cap F^-(V_2).$$

Then $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. By (1), there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_1))$ and

$$\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_2)) \cap F(z) \neq \emptyset$$

for each $z \in U$. Therefore,

$$U \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_1))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_2))).$$

Thus, $x \in \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_1))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_2))))$ and hence

$$\begin{aligned}&F^+(V_1) \cap F^-(V_2)\\ &\subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_1))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_2)))).\end{aligned}$$

(2) $\Rightarrow$ (3): Let K_1, K_2 be any $\sigma_1\sigma_2$-closed sets of Y. Then $Y - K_1$ and $Y - K_2$ are $\sigma_1\sigma_2$-open sets of Y and by (2), we have

$$\begin{aligned}&X - (F^-(K_1) \cup F^+(K_2))\\ &= F^+(Y - K_1) \cap F^-(Y - K_2)\\ &\subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K_1))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K_2))))\\ &= X - \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K_1))) \cup F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K_2))))\end{aligned}$$

and hence

$$\begin{aligned}&\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K_1))) \cup F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K_2))))\\ &\subseteq F^-(K_1) \cup F^+(K_2).\end{aligned}$$

(3) $\Rightarrow$ (4): Let B_1, B_2 be any subsets of Y. Then $\sigma_1\sigma_2\text{-Cl}(B_1)$ and $\sigma_1\sigma_2\text{-Cl}(B_2)$ are $\sigma_1\sigma_2$-closed in Y and by (3),

$$\begin{aligned}&\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B_1)))) \cup F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B_2)))))\\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B_1)) \cup F^+(\sigma_1\sigma_2\text{-Cl}(B_2)).\end{aligned}$$

(4) $\Rightarrow$ (5): Let B_1, B_2 be any subsets of Y. By (4), we have

$$\begin{aligned}&F^-(\sigma_1\sigma_2\text{-Int}(B_1)) \cap F^+(\sigma_1\sigma_2\text{-Int}(B_2))\\ &= X - (F^+(\sigma_1\sigma_2\text{-Cl}(Y - B_1)) \cup F^-(\sigma_1\sigma_2\text{-Cl}(Y - B_2)))\\ &\subseteq X - \tau_1\tau_2\text{-Cl}[F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B_1))))\\ &\quad \cup F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B_2))))]\\ &= \tau_1\tau_2\text{-Int}[F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B_1))))\\ &\quad \cap F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B_2))))].\end{aligned}$$

(5) ⇒ (2): The proof is obvious.
(2) ⇒ (1): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y such that

$$x \in F^+(V_1) \cap F^-(V_2).$$

By (2), we have

$$\begin{aligned} &x \in F^+(V_1) \cap F^-(V_2) \\ &\subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_1))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_2)))). \end{aligned}$$

Then, there exists a $\tau_1\tau_2$-open set U of X such that

$$x \in U \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_1))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_2))).$$

Thus, $F(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_1))$ and $F(z) \cap \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_2)) \neq \emptyset$ for every $z \in U$. This shows that F is almost (τ_1, τ_2)-continuous. □

Definition 3.3. [2] *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be almost* (τ_1, τ_2)*-continuous at a point* $x \in X$ *if for and each* $\sigma_1\sigma_2$*-open set* V *of* Y *containing* $f(x)$*, there exists a* $\tau_1\tau_2$*-open set* U *of* X *containing* x *such that* $f(U) \subseteq \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))$*. A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be almost* (τ_1, τ_2)*-continuous if* f *has this property at each point of* X*.*

Corollary 3.4. [2] *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is almost* (τ_1, τ_2)*-continuous;*

(2) $f^{-1}(V) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq f^{-1}(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(5) $f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B)))))$ *for every subset* B *of* Y*.*

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