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(τ_1, τ_2)-continuity and $(\tau_1, \tau_2)\theta$-closed sets

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Abstract

In this paper, we characterize upper and lower (τ_1, τ_2)-continuous multifunctions by utilizing the notions of $(\tau_1, \tau_2)\theta$-closed sets and $(\tau_1, \tau_2)\theta$-open sets.

1 Introduction

Topology deals with all questions directly or indirectly related to continuity. Semi-open sets, preopen sets, α-open sets and β-open sets play important roles in generalizations of continuity. Using these sets, many authors introduced and investigated various types of weak forms of continuity for functions

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and multifunctions. In 1993, Popa and Noiri [9] obtained some characterizations of upper and lower α-continuous multifunctions. Moreover, Popa and Noiri [8] introduced and studied the notions of upper and lower β-continuous multifunctions. In 2000, Noiri and Popa [7] investigated the concepts upper and lower M-continuous multifunctions as multifunctions defined between sets satisfying certain minimal conditions. Popa and Noiri [10] introduced and studied the notion of m-continuous multifunctions. Laprom et al. [6] introduced and studied the notions of upper and lower $\beta(\tau_1, \tau_2)$-continuous multifunctions. Viriyapong and Boonpok [12] introduced and investigated the concepts of upper and lower $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Furthermore, several characterizations of $(\tau_1, \tau_2)\delta$-semicontinuous mutifunctions and almost weakly (τ_1, τ_2)-continuous multifunctions were investigated in [4] and [3], respectively. In this paper, we investigate several characterizations of upper and lower (τ_1, τ_2)-continuous multifunctions by utilizing the notions of $(\tau_1, \tau_2)\theta$-closed sets and $(\tau_1, \tau_2)\theta$-open sets.

2 Preliminaries

Throughout this paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [5] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [5] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [5] of A and is denoted by $\tau_1\tau_2$-Int(A). Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$-*cluster point* [12] of A if $\tau_1\tau_2\text{-Cl}(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$-*closure* [12] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-*closed* [12] if $(\tau_1, \tau_2)\theta\text{-Cl}(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$-*open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$-*interior* [12] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3 Characterizations of upper and lower (τ_1, τ_2)-continuous multifunctions

In this section, we investigate some characterizations of upper and lower (τ_1, τ_2)-continuous multifunctions.

Definition 3.1. [11] *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:*

(1) upper (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$;

(2) lower (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$.

Lemma 3.2. [11] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper (τ_1, τ_2)-continuous;

(2) $F^+(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^-(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-$Cl(F^-(B)) \subseteq F^-(\sigma_1\sigma_2$-$Cl(B))$ for every subset B of Y;

(5) $F^+(\sigma_1\sigma_2$-$Int(B)) \subseteq \tau_1\tau_2$-$Int(F^+(B))$ for every subset B of Y.

Definition 3.3. *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-regular if for each $\tau_1\tau_2$-closed set F and each $x \notin F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Lemma 3.4. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-regular if and only if for each $x \in X$ and each $\tau_1\tau_2$-open set U containing x, there exists a $\tau_1\tau_2$-open set V such that $x \in V \subseteq \tau_1\tau_2$-$Cl(V) \subseteq U$.*

Lemma 3.5. *Let (X, τ_1, τ_2) be a (τ_1, τ_2)-regular space. Then, the following properties hold:*

(1) $\tau_1\tau_2$-Cl$(A) = (\tau_1, \tau_2)\theta$-Cl$(A)$ for every subset A of X.

(2) Every $\tau_1\tau_2$-open set is $(\tau_1, \tau_2)\theta$-open.

Proof. (1) In general, we have $\tau_1\tau_2\text{-Cl}(A) \subseteq (\tau_1, \tau_2)\theta\text{-Cl}(A)$ for every subset A of X. Next, we show that $(\tau_1, \tau_2)\theta\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(A)$. Let

$$x \in (\tau_1, \tau_2)\theta\text{-Cl}(A)$$

and U be any $\tau_1\tau_2$-open set of X containing x. By Lemma 3.4, there exists a $\tau_1\tau_2$-open set V such that $x \in V \subseteq \tau_1\tau_2\text{-Cl}(V) \subseteq U$. Since

$$x \in (\tau_1, \tau_2)\theta\text{-Cl}(A),$$

it follows that $\tau_1\tau_2\text{-Cl}(V) \cap A \neq \emptyset$ and hence $U \cap A \neq \emptyset$. Thus, $x \in \tau_1\tau_2\text{-Cl}(A)$ and so $(\tau_1, \tau_2)\theta\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(A)$.

(2) Let V be a $\tau_1\tau_2$-open set. By (1), we have

$$X - V = \tau_1\tau_2\text{-Cl}(X - V) = (\tau_1, \tau_2)\theta\text{-Cl}(X - V)$$

and hence $X - V$ is $(\tau_1, \tau_2)\theta$-closed. Thus, V is $(\tau_1, \tau_2)\theta$-open. □

Theorem 3.6. *Let (Y, σ_1, σ_2) be a (σ_1, σ_2)-regular space. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper (τ_1, τ_2)-continuous;

(2) $F^-((\sigma_1, \sigma_2)\theta$-Cl$(B))$ is $\tau_1\tau_2$-closed in X for every subset B of Y;

(3) $F^-(K)$ is $\tau_1\tau_2$-closed in X for every $(\sigma_1, \sigma_2)\theta$-closed set K of Y;

(4) $F^+(V)$ is $\tau_1\tau_2$-open in X for every $(\sigma_1, \sigma_2)\theta$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. By Lemma 3.5, $(\sigma_1, \sigma_2)\theta\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y. Since F is upper (τ_1, τ_2)-continuous, by Lemma 3.2

$$F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$$

is $\tau_1\tau_2$-closed in X.

(2) $\Rightarrow$ (3): Let K be any $(\sigma_1, \sigma_2)\theta$-closed set of Y. Then, $(\sigma_1, \sigma_2)\theta\text{-Cl}(K) = K$ and by (2), we have $F^-(K)$ is $\tau_1\tau_2$-closed in X.

(3) $\Rightarrow$ (4): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for any subset B of Y.

(4) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, we have V is $(\sigma_1, \sigma_2)\theta$-open in Y and by (4), $F^+(V)$ is $\tau_1\tau_2$-open in X. Thus, F is upper (τ_1, τ_2)-continuous by Lemma 3.2. □

Theorem 3.7. *Let (Y, σ_1, σ_2) be a (σ_1, σ_2)-regular space. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower (τ_1, τ_2)-continuous;

(2) $F^+((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ is $\tau_1\tau_2$-closed in X for every subset B of Y;

(3) $F^+(K)$ is $\tau_1\tau_2$-closed in X for every $(\sigma_1, \sigma_2)\theta$-closed set K of Y;

(4) $F^-(V)$ is $\tau_1\tau_2$-open in X for every $(\sigma_1, \sigma_2)\theta$-open set V of Y.

Proof. The proof is similar to that of Theorem 3.6. □

Definition 3.8. [2] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous if f has this property at each point of X.*

Corollary 3.9. *Let (Y, σ_1, σ_2) be a (σ_1, σ_2)-regular space. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is (τ_1, τ_2)-continuous;

(2) $f^{-1}((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ is $\tau_1\tau_2$-closed in X for every subset B of Y;

(3) $f^{-1}(K)$ is $\tau_1\tau_2$-closed in X for every $(\sigma_1, \sigma_2)\theta$-closed set K of Y;

(4) $f^{-1}(V)$ is $\tau_1\tau_2$-open in X for every $(\sigma_1, \sigma_2)\theta$-open set V of Y.

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