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Two concepts of compactness that characterize the collection of Borel sets of $\mathbb{R}$

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Abstract

In this paper we present two concepts of compactness. The first one is related with inf-compact subset of any interval $(a, b]$, while the second one deals with sup-compact subset of any interval $[a, b)$. The structure of these sets allows us to reach different conclusions about both their cardinality and measurement. In the end, we focus on the relation between generalized compact sets and Borel sets on $\mathbb{R}$. We notice that the generalized compact sets presented in the material fully characterize this collection, meaning that each of them generates it.

1 Introduction

Borel sets play a crucial role in measure theory, since any measure defined on either open or closed sets of a space must also be defined on all Borel sets of that space. The notion of Borel sets finds its origin in the research conducted by Baire in 1899. Borel defined these sets for the first time in his paper entitled "Leçons sur les fonctions de réelles" in 1905 [6]. Since

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then, Borel sets are used in various fields of mathematics and beyond. For example, we mention the work of Choquet in potential theory. The theory of Borel sets has found uses in diverse applied areas such as optimization, control theory, mathematical economics and mathematical statistics [2].
Recently, these sets have been used in Quantum Mechanics. It is shown in Section 1 of [5] that for quantum experiments the different measured σ-algebras cannot all be included into a single σ-algebra. On the other hand, in classical physics, the measured σ-algebras all sit inside the σ-algebra $B(\Omega)$ of essential properties of the system, consisting of the σ-algebra generated by the open sets of the phase space Ω of the system.
Our paper is initially focused on two concepts of compactness, one of which is in section 2.1 of [3], where the inf-compact set in $(0, 1]$ is defined. Here, we define inf-compact sets in any interval $(a, b]$ and sup-compact sets in any interval $[a, b)$. In section 3.1, we present the structure and some properties of these sets such as:
A bounded subset $K \subseteq \mathbb{R}$ is compact if and only if it is simultaneously an inf-compact subset of $(a, b]$ and a sup-compact subset of $[a, b)$, where $a = inf(K)$ and $b = sup(K)$.
Considering this, these two concepts of compactness are generalizations of the known concept of compactness.
Cardinality and measurasbility of inf-compact and sup-compact collections are the first issues discussed in section 3.2. To each finite set A we can assign, in principle, a number called the cardinal number or cardinality of A. Practically, in case of infinite sets, to find this number might be difficult. There are used common symbols for the cardinal number associated to the most used sets. So $\aleph_0$ is the cardinal number for every countable set. Cantor chose "c" for $| \mathbb{R} |$ since is the first letter of the word "continuum".
For example, We can write $| [0, 1] |=| (0, 1) |=| \mathbb{R} |= c$ ([2] (section 12)).
We have reached the conclusion that these collections have the cardinality of the continuum. Starting from the construction of inf-compact and sup-compact subsets of $(a, b]$ and $[a, b)$ respectively, we conclude that these sets are part of the collection of Borel sets (see [1], proposition 1.1.4). So we have noticed that the constructed collections are included in the Borel collection sets in $\mathbb{R}$ and have the same cardinality with it [4]. The last conclusion gave us the idea to analyze if the collection of Borel sets in $\mathbb{R}$ could be generated from them. The validity of this expectation represents the main conclusion of our study and allows to use the Borel set in fuzzy sets theory (see [3], section 2.1 for future studies).

2 Preliminaries

Definition 2.1. *[2] (definition 7.1)*
The sets A and B are equivalent, written $A \sim B$, if there exists a bijection $f : A \to B$.

Definition 2.2. *[2] (definition 7.5)*
The set A is called countable if there exists a one-to-one map $f : A \to \mathbb{N}$.

The set of rational numbers $\mathbb{Q}$ is countable ([2] (example 7.9)).
Every set that is not countable is called uncountable. The interval of real numbers $(0, 1)$ is uncountable ([2] (theorem 7.7)). The set $\mathbb{R}$ of real numbers and the set of irrational numbers $\mathbb{I}$ are both uncountable ([2] (example 7.9)).
In set theory, each finite set A can be associated with a number called the cardinal number, representing the size or cardinality of A. It answers the question: "How many elements does A have?". The symbol $|A|$ represents the cardinal number of A. Practically, in case of infinite sets, to find this number might be difficult. To make this more specific, we will assume that to each set there is associated an "object" denoted by $|A|$, which is called cardinal number of A, and this is done in such a way that $|A|= |B|$ if and only if $A \sim B$. There are used common symbols for the cardinal number associated to the most used sets. So, $\aleph_0$ is the cardinal number for every countable set. Cantor chose "c" for $|\mathbb{R}|$ since is the first letter of the word "continuum". For example, we can write $|[0, 1]|= |(0, 1)|= |\mathbb{R}|= c$ ([2] (section 12)).
Defining an order " $\leq$ " between cardinal numbers allows us for a more precise characterization. Let's suppose that the sets A and B have cardinal numbers m and m respectively.

Definition 2.3. *[2] (definition 13.1)*

1) $m \leq n$ means that A is equivalent to a subset of B.(We also write $m \leq n$ as $n \geq m$.)
2) $m < n$ (or $n > m$) means that $m \leq n$ but $m \neq n$. (We also write $m < n$ as $n > m$.)

If $m \leq n$ and $n \leq m$, then $m = n$ ([2] (theorem 13.2 (3)).
The cardinal of collection $\mathcal{P}(\mathcal{A})$ of all subsets of set A is denoted with 2^m.From [2] (example 14.2) we have $m < 2^m$.
Let X be a non-empty set.

Definition 2.4. *[1] (section 1.1)*

Let Σ be a collection of subsets of X. We call Σ a σ - algebra of subsets of X if it is non-empty, closed under complements and closed under countable unions. This means that:
(i) there exists at least one $A \subseteq X$ so that $A \in \Sigma$,
(ii) if $A \in \Sigma$, then $A^c \in \Sigma$, where $A^c = X \setminus A$, and
(iii) if $A_n \in \Sigma$ for all $n \in N$, then $(\cup_{n=1}^{+\infty} A_n) \in \Sigma$.
The pair (X, Σ) of a non-empty set X and a σ - algebra Σ of subsets of X is called a measurable space.

Definition 2.5. *[1](section 1.1)*

The intersection of all σ - algebras that contains a collection is called the smallest σ - algebra that contains it. In this case, we say that this σ - algebra is generated from the mentioned collection.

The Borel σ-algebra on $\mathbb{R}$ is generated by the collection of open subsets of $\mathbb{R}$. It is denoted by $\mathcal{B}(\mathbb{R})$. Elements of this σ - algebra are called Borel sets ([1](section 1.1)).

Definition 2.6. *[1] (section 1.2)*

Let (X, Σ) be a measurable space. A function $\mu : \Sigma \to [0, +\infty]$ is called a measure on (X, Σ) if
(i) $\mu(\emptyset) = 0$
(ii)$\mu(\cup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty} \mu(A_n)$ for all sequences (A_n) of pairwise disjoint sets which are contained in Σ.

Let X be a set and let $\mathcal{P}(X)$ be the collection of all subsets of X. An outer measure on X is a function $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ such that ([1](section 1.3)):
(a) $\mu^*(\emptyset) = 0$,
(b) if $A \subseteq B \subseteq X$, then $\mu^*(A) \leq \mu^*(B)$, and
(c) if $(A_n)_{n\in\mathbb{N}}$ is a sequence of subsets of X, then $\mu^*(\cup_n A_n) \leq \Sigma_n \mu^*(A_n)$.

Definition 2.7. *[1](example 1.3.1 (d))*

Lebesgue outer measure on $\mathbb{R}$, which we will denote by λ^*, is defined as follows:
For each subset A of $\mathbb{R}$, let $\mathcal{C}_A$ be the set of all infinite sequences $\{(a_i, b_i)\}$ of bounded open intervals such that $A \subseteq \cup_{i=1}^{\infty}(a_i, b_i)$. Then $\lambda^* : \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ is defined by $\lambda^*(A) = inf\{\sum_i (b_i - a_i) : \{(a_i, b_i)\} \in \mathcal{C}_A\}$.

Lebesgue outer measure on $\mathbb{R}$ is an outer measure and it assigns to each subinterval of $\mathbb{R}$ its length ([1](proposition 1.3.2).
Let X be a set and let μ^* be an outer measure on X. A subset B of X is μ^*- measurable (or measurable with respect to μ^*) if $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$ holds for every subset A of X. Every subset B of $\mathbb{R}$ that is λ^*-meaurable called Lebesgue measurable. Every Borel subset of $\mathbb{R}$ is Lebesgue measurable ([1] (proposition 1.3.7).
The triple (X, Σ, μ) of a non-empty set X, a σ - algebra of subsets of X and a measure μ on Σ is called a measure space. The measure μ (or the measure space (X, Σ, μ) is complete if the relations $A \in \Sigma, \mu(A) = 0$, and $B \subseteq A$ together imply that $B \in \Sigma$. Lebesgue measure on the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$ is complete. On the other hand, the restriction of Lebesgue measure to the σ-algebra of Borel subsets of $\mathbb{R}$ is not complete, [1], section 1.5.

Definition 2.8. *([3], section 2.1)*
A subset $L \subseteq (0,1]$ *such that there exists Min(*$[\alpha, 1] \cap L$*), for any* $\alpha \in (0,1]$, *is called an inf-compact subset of* $(0,1]$.

3 Main results

In this section we show the structure of inf-compact and sup-compact sets and some of their properties. We give some examples of these types of sets and what happens when we add a countable amount of points to the inf-compact subset of $(a, b]$. We conclude that these sets are part of the collection of Borel sets and, as above, determine the following definitions:

Definition 3.1. *(i)A subset* $L \subseteq (a, b]$ *such that there exists Min(*$[\alpha, b] \cap L$*), for any* $\alpha \in (a, b]$ *is called an inf-compact subset of (a,b].*
(ii) A subset $L \subseteq [a, b)$ *such that there exists Max(*$[a, \alpha] \cap L$*) for any* $\alpha \in [a, b)$ *is called an sup-compact subset of [a,b).*

3.1 The structure of inf-compact and sup-compact sets and some of their properties

Let $L \subseteq (a, b]$ be an inf-compact subset of $(a, b]$. Based on the definition of inf-compact sets, we can easily see that the following lemma holds:

Lemma 3.2. *If $x \notin L$, then exists an $\alpha \in L$ such that $[x, \alpha) \subset (a, b] \setminus L$.*

Proposition 3.3. *If the smallest element of A is found, then we find a number a such that $[a, \alpha) \subset (a, b] \setminus L$ where $a = Min(A)$. Otherwise, we find a number a such that $(a, \alpha) \subset (a, b] \setminus L$, where $a = inf(A)$.*

Proof.

If the $Min(A) = a$ exists, then $y \notin L$ for every $y \in [a, \alpha)$.
Suppose that the $Min(A)$ does not exist. Denote $a = inf(A)$ and for every $\varepsilon > 0$ exists $x_1 \in A$ such that $a \leq x_1 < a + \varepsilon$. So, $a < x_1 < y = a + \varepsilon$ and $Min([x_1, b] \cap L) = \alpha$. Since ε is a very small positive number, $y < \alpha$ and $([y, b] \cap L) \subset ([x_1, b] \cap L)$. Thus $Min([y, b] \cap L) = \alpha$ and $y \notin L$. So we can find an element y that is not part of L even though y tends to a. As a result, $(a, \alpha) \subset (a, b] \setminus L$.

Proposition 3.4. *Every two different intervals of type 1 do not have common points.*

Proof.

If x is a common point of two different intervals, with right endpoints respectively α and β , then

$$Min([x, b] \cap L) = \alpha$$

and also

$$Min([x, b] \cap L) = \beta$$

But this is impossible. This completes the proof.

Some examples of inf-compact sets in $(0, 1]$ are given in [?]. So any interval $[\alpha, \beta) \subseteq (0, 1]$, union with $\{1\}$ is inf-compact set. Also, any ascending sequence in (0,1] union with $\{1\}$, is inf-compact set. In the first case, we see that $L = [\alpha, \beta) \cup \{1\} = (0, 1] \setminus ((0, \alpha) \cup [\beta, 1))$. In the second case, we see that the set L of terms of ascending sequence in $(0, 1]$, union with $\{1\}$ is taken from $(0, 1]$ leaving a countable amount of intervals of type 1.

Do these two examples give us the construction method of every inf-compact set?
The following proposition gives us the answer of this question.

Proposition 3.5. *Every inf-compact subset of $(a, b]$ is taken from $(a, b]$ leaving a finite or countable amount of intervals of type 1.*

Proof.
In the examples above, we saw that there are inf-compact sets that are obtained from $(0,1]$ *leaving a finite amount of intervals of type 1. One of them is* $L = [\alpha, \beta) \cup \{1\}$.
Let us show that the number of intervals of type 1 that leaves from $(a,b]$ *for constructing an inf-compact set can not be more than countable.*
It is well-known that the set of rational numbers in $\mathbb{R}$ *is a countable set. So, we can arrange all its elements as a numerical sequence. Let* $r_1, r_2, ..., r_n, ...$ *be the sequence of all rational numbers in* $\mathbb{R}$*. Let* A *be the set of all intervals of type 1 that leaves from* $(a,b]$ *for constructing the inf-compact set* L*. Let's consider the function* $f : A \to \mathbb{Q}$*, such that: every interval of type 1 corresponds to the first rational number in the above sequence that is part of this interval. Since the intervals have no common points, this function is injective. So* $card(A) \leq card(\mathbb{Q}) = \aleph_0$*, where* $\aleph_0$ *is denoted as the cardinal of numerable sets. This completes the proof.*

Every finite union of inf-compact subsets of $(0,1]$ is also an inf-compact subset of $(0,1]$. Every finite subset of $(0,1]$ containig 1 is an inf-compact subset of $(0,1]$ ([3], section 2.1).
So we conclude that if we add a finite number of points to an inf-compact subset of $(a,b]$, then the obtained set is again an inf-compact subset of $(a,b]$. The above conclusion does not hold if we add a countable amount of points to the inf-compact subset of $(a,b]$.

Lemma 3.6. *If* L *is an inf-compact subset of* $(a,b]$ *such that* L *is taken from* $(a,b]$ *leaving a countable amount of intervals of type* 1*, then we can add in* L *a countable amount of points, respectively from one to each of the intervals* δ_n *of type* 1*, and the obtained set is again an inf-compact subset of* $(a,b]$.

The following proposition holds.

Proposition 3.7. *If* $(L_n)_{n\in\mathbb{N}}$ *is a sequence of inf-compact subsets of* $(a,b]$*, then* $\cap_{n\in\mathbb{N}} L_n$ *is an inf-compact subset of* $(a,b]$.
Proof.
A similar result has been proven in [3] Lemma 1, for inf-compact subsets of $(0,1]$*. We will prove it by using the inf-compact set structure.*
Every $L_n = (a,b] \setminus (\cup_{i\in I_n} \delta_i)$*, where* I_n *is a finite or countable set. So, we can write:*
$\cap_{n\in\mathbb{N}} L_n = \cap_{n\in\mathbb{N}}((a,b] \setminus (\cup_{i\in I_n} \delta_i)) = \cap_{n\in\mathbb{N}}((a,b] \cap (\cup_{i\in I_n} \delta_i)^c) = \cap_{n\in\mathbb{N}}((a,b] \cap (\cap_{i\in I_n} \delta_i^c))) = (a,b] \cap (\cap_{n\in\mathbb{N}}(\cap_{i\in I_n} \delta_i^c)) = (a,b] \cap (\cap_{n\in\mathbb{N}}(\cup_{i\in I_n} \delta_i)^c) = (a,b] \cap$

$(\cup_{n\in\mathbb{N}}(\cup_{i\in I_n}\delta_i))^c = (a, b] \setminus (\cup_{n\in\mathbb{N}}(\cup_{i\in I_n}\delta_i))$.
The collection $\{\delta_i : i \in I_n, n \in \mathbb{N}\}$ *is a countable collection of intervals of type* 1*. This completes the proof.*

Let $L' \subseteq [a, b)$ be a sup-compact subset of $[a, b)$. Based on the definition of sup-compact sets, we can easily see that the following lemma holds.

Lemma 3.8. *If* $y \notin L'$*, then exists a* $\beta \in L'$ *such that* $(\beta, y] \subset [a, b) \setminus L'$.

Clearly, for every $z \in (\beta, y]$, we can also write $(\beta, z] \subset (a, b] \setminus L'$.
Let denote $B = \{y \in [a, b) \setminus L' : Max([a, y] \cap L') = \beta)\}$. We can proved the following proposition.

Proposition 3.9. *If the largest element of* B *exists, then we find a number* b *such that* $(\beta, b] \subset [a, b) \setminus L'$ *where* $b = Max(B)$*. Otherwise, we find a number* b *such that* $(\beta, b) \subset [a, b) \setminus L'$*, where* $b = sup(B)$.

The proof closely follows that of proposition 3.1 with the distinction that here we use the supremum property.
The intervals mentioned in the above proposition are called the intervals of type 2. Two intervals are considered different when they have different left endpoints.
As in the proposition 3.2, we have proposition:

Proposition 3.10. *Every two different intervals of type* 2 *do not have common points.*

We can easily see that any interval $(\alpha, \beta] \subseteq [a, b)$ union with $\{a\}$, is sup-compact set. Also, any descending sequence $L' = (x_n)_{n\in\mathbb{N}}$ in $[a, b)$ union with $\{a\}$ is sup-compact set.
Let's give the reason of that. Let's give the explanation for that:
If $[a, \alpha] \cap L' = \{a\}$, then $Max([a, \alpha] \cap L') = a$. If there is any $x \neq a$ in $[a, \alpha] \cap L'$, then $x = x_{n_0}$ and $x > x_n$ for every $n > N$. So, $Max([a, \alpha] \cap L') = Max\{x_n : x_n \leq \alpha, n = n_1, n_2, ..., N\}$.
In the first case, we see that $L' = (\alpha, \beta] \cup \{a\} = [a, b) \setminus ((a, \alpha] \cup (\beta, b))$. In the second case, we see that the set L' of terms of descending sequence in $[a, b)$, union with $\{a\}$ is taken from $[a, b)$ leaving a countable amount of intervals of type 2.
In a way similar to that of proposition 3.3, we have proved the following proposition.

Proposition 3.11. *Every sup-compact subset of* $[a, b)$ *is taken from* $[a, b)$ *leaving a finite or countable amount of intervals of type* 2.

Since every compact set in $\mathbb{R}$ is closed and bounded, and also based on the construction of the open sets in $\mathbb{R}$, we reach the following conclusion that a bounded subset $K \subseteq \mathbb{R}$ is compact if and only if it is simultaneously to an inf-compact subset of $(a, b]$ and to a sup-compact subset of $[a, b)$, where $a = inf(K)$ and $b = sup(K)$.

The last conclusion allows us to call inf-compact and sup-compact sets generalized compact sets.

In a way similar to inf-compact subsets of $(a, b]$, we observe the following conclusions
a) Every finite union of sup-compact subsets of $[a, b)$ is also a sup-compact subset of $[a, b)$. Every finite subset of $[a, b)$ containing a is a sup-compact subset of $[a, b)$.
b) If we add a finite number of points to a sup-compact subset of $[a, b)$, then the obtained set is again sup-compact subset of $[a, b)$
c) If L' be an sup-compact subset of $[a, b)$ such that L is taken from $[a, b)$ leaving a countable amount of intervals of type 2, then we can add in L' a countable amount point, respectively from one to each of the intervals δ_n of type 2, and the obtained set is again sup-compact subset of $[a, b)$.
d) If $(L'_n)_{n\in\mathbb{N}}$ is a sequence of sup-compact subsets of $[a, b)$, then $\cap_{n\in\mathbb{N}} L'_n$ is sup-compact subset of $[a, b)$.

3.2 Cardinality, measurasbility and the relation between generalized compact sets and Borel sets on $\mathbb{R}$

Starting from the construction of inf-compact and sup-compact subsets respectively of $(a, b]$ and $[a, b)$, we have concluded that these sets are part of the collection of Borel sets (see [1], proposition 1.1.4). Denote $\mathcal{I}(C)$, $\mathcal{S}(C)$ the collection of all inf-compact subsets of any interval $(a, b]$ and sup-compact subsets of any interval $[a, b)$ respectively. So the collections $\mathcal{I}(C)$ and $\mathcal{S}(C)$ are Borel measurable (therefore also Lebesgue measurable [1], proposition 1.3.7). The following proposition holds:

Proposition 3.12. *The collections $\mathcal{I}(C)$ and $\mathcal{S}(C)$ have the cardinality of continuum.*

Proof.

As we stated before, $\mathcal{I}(C) \subset \mathcal{B}(R)$ and $\mathcal{S}(C) \subset \mathcal{B}(R)$. Also, the collec-

tion of Borel set in $\mathbb{R}$ has the cardinal of continuum (see [?]). So it is true that $|\mathcal{I}(C)| \leq |\mathcal{B}(R)| = c$ and $|\mathcal{S}(C)| \leq |\mathcal{B}(R)| = c$. Since every interval $[\alpha, \beta) \subset (a, b]$ union with $\{b\}$ is a inf-compact subset of $(a, b]$ and every interval $(\alpha, \beta] \subset [a, b)$ union with $\{a\}$ is a sup-compact subset of $[a, b)$, we considered bijective functions $f : \{(\alpha, \beta] \cup \{b\}\} \to (a, b)$ and $g : \{[\alpha, \beta) \cup \{a\}\} \to (a, b)$ such that $f((\alpha, \beta] \cup \{b\}) = \beta$ and $g([\alpha, \beta) \cup \{a\}) = \alpha$. Therefore $c = |(a, b)| \leq |\mathcal{I}(C)| \leq |\mathcal{B}(R)| = c$ and $c = |(a, b)| \leq |\mathcal{S}(C)| \leq |\mathcal{B}(R)| = c$. We are in the same conditions as theorem 13.2(3) [?] and we write that the collections $\mathcal{I}(C)$ and $\mathcal{S}(C)$ have the cardinality of continuum.

Every interval (a, b) is neither an inf-compact subset of $(a, b]$ nor a sup-compact subset of $[a, b)$. So we can find Borel sets that are not in $\mathcal{I}(C)$, $\mathcal{S}(C)$. Therefore, the collections $\mathcal{I}(C)$ and $\mathcal{S}(C)$ are included in collections of Borel sets.
Denote by $\sigma(\mathcal{I}(C))$ and $\sigma(\mathcal{S}(C))$ the smallest σ-algebra that contains $\mathcal{I}(C)$ and $\mathcal{S}(C)$ collections respectively.
The Cantor set C is a compact set that has the cardinality of continuum but has Lebesgue measure zero ([?], example 1.4.6). Therefore, $C \in \sigma(\mathcal{I}(C))$ and $C \in \sigma(\mathcal{S}(C))$, as well. On the other hand, we can find subsets of C that are not inf-compact subsets of $(0, 1]$ or sup-compact subset of $[0, 1)$. Since the cardinal of collection of subsets of Cantor set is 2^c and the collections $\mathcal{I}(C)$ and $\mathcal{S}(C)$ have the cardinality of continuum, we can find subsets of Cantor set which are not part of these collections. So the measure spaces $(\mathbb{R}, \sigma(\mathcal{I}(C)), \lambda)$ and $(\mathbb{R}, \sigma(\mathcal{S}(C)), \lambda)$, where λ is the restriction of Lebesgue measure on $\mathcal{I}(C)$ and $\mathcal{S}(C)$ collections are not complete.
Moreover, the following proposition holds.

Proposition 3.13. *The $\sigma(\mathcal{I}(C))$ and $\sigma(\mathcal{S}(C))$ σ-algebras are the same as the Borel σ-algebra.*
Proof.
As we mentioned before, the $\mathcal{I}(C)$, $\mathcal{S}(C)$ collections are part of Borel sets. So the $\sigma(\mathcal{I}(C))$ and $\sigma(\mathcal{S}(C))$ σ-algebras are included to Borel σ-algebra.
Every interval $(a, b]$ is a Borel set, [1] proposition 1.1.4, and it is an inf-compact subset of $(a, b]$, as well.
Every interval (a, b) is a Borel set because it is an open set in $\mathbb{R}$. Moreover, we can write $[a, b) = \cap_{n \in \mathbb{N}} (a - \frac{1}{n}, b)$. So, every $[a, b)$ is a Borel set and it is a sup-compact set of $[a, b)$, as well.
Therefore, the Borel σ-algebra is included in both of $\sigma(\mathcal{I}(C))$ and $\sigma(\mathcal{S}(C))$ σ-algebras.

4 Conclusions

Below, we have ordered some conclusions regarding to our research:

The σ-algebra $\mathcal{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$ is generated by each of two following collections of sets:
1. the collection of all inf-compact subsets of any interval $(a, b]$,
2. the collection of all sup-compact subsets of any interval $[a, b)$.

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