International Journal of Mathematics and
Computer Science, **19**(2024), no. 4, 1093–1099

Structure of Pseudo Near Semirings

Sawanya Jadpimai, Utsanee Leerawat

Department of Mathematics
Faculty of Science
Kasetsart University
Bangkok 10900, Thailand

email: sawanya.jad@ku.th, fsciutl@ku.ac.th

(Received March 3, 2024, Accepted April 4, 2024,
Published June 1, 2024)

Abstract

In this paper, we introduce a pseudo near semiring (PNS) which is a novel generalization of additive inverse semirings and near semirings. Moreover, we investigate some interesting algebraic properties of PNSs including the notion of additive idempotents. Furthermore, we provide some results on pseudo inverse elements.

1 Introduction

A natural generalization of rings, semirings were first introduced by Vandiver [18] in 1934. Vandiver defined a semiring as an algebraic system consisting of a nonempty set S equipped with two binary operations, addition $(+)$ and multiplication $(\cdot)$, such that $(S,+)$ and $(S,\cdot)$ are semigroups, satisfying $z\cdot(x+y)=z\cdot x+z\cdot y$ and $(x+y)\cdot z=x\cdot z+y\cdot z$ for all $x,y,z\in N$. A subset T of S is called a subsemiring of S if it is also a semiring under the same binary operations.

Karvellas [12] studied the notion of additive inverse semiring. A semiring S is called an additive inverse semiring if for every $a\in S$ there exists a unique element $a'\in S$, called the additive pseudo inverse of a such that

Key words and phrases: semirings, near semirings, additive inverse semiring.
AMS (MOS) Subject Classifications: 16Y60,16Y30,16Y99.
The corresponding author is Utsanee Leerawat.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

$a+a'+a = a$ and $a'+a+a' = a'$. Bandlet and Petrich [4] characterized inverse semirings as a subdirect product of rings and distributive lattices. Javed and Aslam [9] introduced the notion of Lie ideals in additive inverse semirings and investigated some commutativity conditions on additive inverse semirings with the help of Lie ideals and derivations. Javed, Aslam, and Hussain [10] introduced the notion of MA-semiring which is an additive inverse semiring satisfying the condition that for each $a \in S$, $a + a'$ is in the center $Z(S)$ of S.

Near semirings were first introduced by van Hoorn and van Rootselaar [8]. By a near semiring, we mean an algebraic structure $(N, +, \cdot)$ such that $(N, +)$ and $(N, \cdot)$ are semigroups, and it satisfies only one (left or right) distribution law; i.e., $z \cdot (x + y) = z \cdot x + z \cdot y$ for all $x, y, z \in N$. Building on this concept, van Hoorn [7] further extended the concept of a Jacobson radical from rings to zero-symmetric near-semirings. Many interesting and elegant properties of these near semirings have been studied by Balakrishnan and Perumal [3], Perumal et al. [14, 15], Sardar and Mukherjee [17], as well as others [1, 2, 5, 13,16]). In addition, near semirings hold significance not only as a natural generalization of semirings and near rings but also for their practical applications in computer science (see [11] and [6]).

In this paper, we introduce a novel algebraic structure called a pseudo near semiring (PNS) which is a generalization of both an additive inverse semiring and a near semiring. We also investigate some interesting algebraic properties of PNSs. We discuss the notion of additive idempotents in PNSs and provide some results on pseudo inverse elements.

2 Main results

In this section, we first review near semirings and then introduce a new generalization called pseudo near semiring.

A left near semiring is a nonempty set N together with two binary operations '+' and '$\cdot$' such that $(N, +)$ and $(N, \cdot)$ are semigroups, satisfying left distribution law $z \cdot (x + y) = z \cdot x + z \cdot y$ for all $x, y, z \in N$.
A right near semiring can be defined in a similar fashion.

Definition 2.1. *A nonempty set N together with two binary operations called addition and multiplication, denoted by $+$ and $\cdot$ respectively, is called a pseudo left near semiring if*

(i.) $(N, +, \cdot)$ *is a left near semiring.*

(ii.) *There exists* $0 \in N$ *such that* $x + 0 = x = 0 + x$ *and* $x \cdot 0 = 0 = 0 \cdot x$ *for all* $x \in N$. *The element* $0 \in N$ *is called the zero element of* N.

(iii.) *For each* $x \in N$, *there exists a unique* $y \in N$ *such that* $x = x + y + x$ *and* $y = y + x + y$ *for all* $x, y \in N$. *The element* $y \in N$ *is called the pseudo inverse of* x, *denoted by* x'.

If $(N, +, \cdot)$ is a right near semiring instead of condition (i), then we call $(N, +, \cdot)$ a pseudo right near semiring.

Unless otherwise specified. by a pseudo near semiring (briefly PNS), we mean only a pseudo left near semiring. We usually write $(N, +, \cdot)$ simply as N.

Now, we give an example of a pseudo near semiring.

Example 2.2. *Let* $N = \{0, a, b, c\}$ *with addition* $(+)$ *and multiplication* $(\cdot)$ *tables defined as follows:*

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	c	0	c

Clearly, N *is a pseudo near semiring.*

Theorem 2.3. *In a pseudo near semiring* N, *the zero element of* N *is unique.*

Proof. The proof is straightforward and hence is omitted. □

Notation: Throughout this paper, we will assume $(N, +, \cdot)$ to be a pseudo near semiring. We denote the zero element of N by 0.

Definition 2.4. *Let* $(N, +, \cdot)$ *be a pseudo near semiring and let* S *be a subset of* N. *Then* S *is called a subpseudo near semiring of* N *if*

(i.) $0 \in S$.

(ii.) $x' \in S$ *for all* $x \in S$.

(iii.) $x + y \in S$ *for all* $x, y \in S$.

(iv.) $xy \in S$ *for all* $x, y \in S$.

The following theorem is easy to prove.

Theorem 2.5. *Let Λ be a nonempty index set and let $\{S_\lambda\}_{\lambda\in\Lambda}$ be a family of subpseudo near semirings of N. Then $\cap_{\lambda\in\Lambda}S_\lambda$ is a subpseudo near semiring of N.*

Definition 2.6. *A pseudo near semiring N is additively commutative if $x+y=y+x$ for all $x,y\in N$.*

Definition 2.7. *Let N be a pseudo near semiring. An element $x\in N$ is called an additive idempotent element if $x=x+x$. A pseudo near semiring N is called an additive idempotent pseudo near semiring if every element of N is additive idempotent.*

For any pseudo near semiring N, we define $I(N)=\{x\in N|x+x=x\}$.

Theorem 2.8. *Let N be a pseudo near semiring. Then*

(i.) $x+x', x'+x\in I(N)$ *for all* $x\in N$.

(ii.) $(x')'=x$ *for all* $x\in N$..

Proof. Let $x\in N$.
(i) $(x+x')+(x+x')=x+(x'+x+x')=x+x'$ and
$(x'+x)+(x'+x)=x'+(x+x'+x)=x'+x$. Hence, $x+x', x'+x\in I(N)$.
(ii) Let $y=x'$. We have $x=x+y+x$ and $y=y+x+y$. By uniqueness the pseudo inverse of y, then $y'=x$. Hence $(x')'=x$. □

Theorem 2.9. *Let N be a pseudo near semiring. Then $I(N)$ is an additively commutative subpseudo near semiring of N.*

Proof. Clearly, $I(N)\subseteq N$ and $0\in I(N)$. Let $x,y\in I(N)$. Let $(x+y)'=z$. We have $(x+y)+z+(x+y)=(x+y)$ and $z+(x+y)+z=z$. Then $(z+x)+y+(z+x)=z+x$ and $(y+z)+x+(y+z)=y+z$. Let $w=z+x$. We have
$(x+y)+w+(x+y)=(x+y)+(z+x)+(x+y)=(x+y)+z+(x+y)=(x+y)$, and
$w+(x+y)+w=(z+x)+(x+y)+(z+x)=(z+x)+y+(z+x)=z+x=w$. This implies that $(x+y)'=w$. Hence $z=w=z+x$. Similarly, let $v=y+z$. Then
$(x+y)+v+(x+y)=(x+y)+(y+z)+(x+y)=(x+y)+z+(x+y)=x+y$ and
$v+(x+y)+v=(y+z)+(x+y)+(y+z)=(y+z)+x+(y+z)=y+z=v$.

This implies that $(x+y)' = v$. Hence $z = v = y + z$. Thus
$z + z = (x+y)' + (x+y)' = w + v = z + x + y + z = z$. Hence $z \in I(N)$.
Since $z + z + z = z + z = z$, $z' = z$. Then, by Theorem 2.8(ii),
$(x+y)' = z = z' = ((x+y)')' = x + y$. This follows that $x + y \in I(N)$.
Next, we will show that $x + y = y + x$. Since
$x, y \in I(N)$,$(x+y) + (y+x) + (x+y) = x + y + x + y = x + y$ and
$(y+x) + (x+y) + (y+x) = y + x + y + x = y + x$. Hence $(x+y)' = y + x$.
Therefore $y + x = (x+y)' = x + y$. So $x + y = y + x$. This shows that $I(N)$ is additively commutative. Clearly, $xy + xy = x(y+y) = xy$. Hence
$xy \in I(N)$. Since $I(N)$ is additively commutative and
$x, x + x' \in I(N), x = x + x' + x = x + x + x' = x + x'$. Then
$x' = x' + x + x' = x' + (x + x') + x' = x' + x'$. This implies that $x' \in I(N)$.
Therefore, $I(N)$ is a subpseudo near semiring of N. □

Theorem 2.10. *Let N be a pseudo near semiring. Then*

(i.) $(x+y)' = y' + x' = x' + y'$ *for all* $x, y \in N$.

(ii.) $(xy)' = xy'$ *for all* $x, y \in N$.

(iii.) $(x'y')' = x'y$ *for all* $x, y \in N$.

Proof. Let $x, y \in N$.
(i) By Theorems 2.8 and 2.9, we have $x + x', x' + x, y + y', y' + y \in I(N)$ and $(y + y') + (x' + x) = (x' + x) + (y + y')$.

$$\begin{aligned} (x+y) + (y'+x') + (x+y) &= x + (y+y') + (x'+x) + y \\ &= x + (x'+x) + (y+y') + y = x + y, \\ \text{and } (y'+x') + (x+y) + (y'+x') &= y' + (x'+x) + (y+y') + x' \\ &= y' + (y+y') + (x'+x) + x' = y' + x'. \end{aligned}$$

Therefore $(x+y)' = y' + x'$. Since $I(N)$ is an additively commutative subpseudo near semiring of N and $x', y' \in I(N)$, $y' + x' = x' + y'$. Hence
$(x+y)' = y' + x' = x' + y'$.
(ii) Since $y = y + y' + y$ and $y' = y' + y + y'$,
$xy + xy' + xy = x(y + y' + y) = xy$ and
$xy' + xy + xy' = x(y' + y + y') = xy'$. Thus $(xy)' = xy'$.
(iii) By (ii) and Theorem 2.8 (ii) $(x'y')' = x'(y')' = x'y$. □

Theorem 2.11. *Let N be a pseudo near semiring and let $x, y \in N$ be such that $x + y = 0$. Then $x = y'$.*

Proof. Let $x, y \in N$ be such that $x + y = 0$. Clearly, $0 = 0' = (x + y)' = y' + x' = x' + y'$ By using Theorem 2.10(i), we get $x = x + 0 = x + x' + y' = x + x' + (x + y) + y' = x + y + y' = y'$. □

Theorem 2.12. *In a pseudo near semiring N, if an element $x \in I(N)$ satisfies the condition $x(x + x') = x + x'$, then $x^n = x$ for all positive integer n.*

Proof. Let $x \in I(N)$ be such that $x(x + x') = x + x'$. Then, by Theorem 2.8, we have $x + x', x' + x \in I(N)$. Since $I(N)$ is an additively commutative subpseudo near semiring of N, $x = x + x' + x = x + x + x' = x + x'$. Hence $x^2 = x(x + x') = x + x' = x$. This implies $x^2 = x$. Therefore, by the principle of mathematical induction, $x^n = x$ for all positive integers n. □

Acknowledgement
This research was supported by the Graduate Student Scholarship from the Faculty of Science, Kasetsart University, Bangkok, Thailand.

References

[1] Javed Ahsan, Seminear-rings characterized by their s-ideals –I, Proc. Japan Acad., **71(A)** (1995), 101103.

[2] Javed Ahsan, Seminear-rings characterized by their s-ideals–II, Proc. Japan Acad., **71(A)** (1995), 111113.

[3] R. Balakrishnan, R. Perumal, Left duo seminear-rings, Scientia Magna, **8,** no. 3, (2012), 115–120.

[4] H.J. Bandelt, M. Petrich, Subdirect products of rings and distributive lattices, Proc. Edinburgh Math. Soc., **25,** (1982), 135–171.

[5] K. Chakraborty, R. Mukherjee, S.K. Sardar, Near-ring congruences on seminearrings, Semigroup Forum, **104,** (2022), 584–593. https://doi.org/10.1007/s00233-021-10249-z

[6] J. Desharnais, G. Struth, Domain axioms for a family of near-semirings, AMAST, (2008), 330–345.

[7] W. G. van Hoorn, Some generalisations of the Jacobson radical for semi-nearrings and semirings, Math. Z., **118,** (1970), 69–82.

[8] W.G. van Hoorn, B. van Rootselaar, Fundamental notions in the theory of semi-nearrings, Compositio Math., **18,** (1967), 65–78.

[9] M.A. Javed, M. Aslam, Some commutativity conditions in prime MA-semirings, ARS Combinatoria, **114,** (2014), 373–384.

[10] M.A. Javed, M. Aslam, M. Hussain, On Condition (A2) of Bandlet and Petrich for inverse semirings, International Mathematical Forum, **7,** (2012), 2903–2914.

[11] C. Jenila, P. Dheena, Ideal theory in near-semirings and its applications to automata, Adv. Math. Sci. J., **9,** (2020), 4293–4302.

[12] P.H. Karvellas, Inverse semirings, J. Aust. Math. Soc., **18,** (1974), 277–288.

[13] R. Mukherjee, S.K. Sardar, P. Pal. On additively commutative near-ring congruences on seminearrings, Semigroup Forum, **91,** no. 3, (2015), 573–583.

[14] R. Perumal, R. Balakrishnan, Left bipotent seminear-rings, International Journal of Algebra, **6,** no. 26, (2012), 1289-1295.

[15] R. Perumal, R. Arulprakasam, M. Radhakrishnan, A note on ideals in seminear-rings, IOP conf. Series, Journal of Physics: Conference Series, 1000, (2018), 012150.

[16] M. Samman, Inverse semigroups and seminear-rings, Journal of Algebra and Its Applications, **8,** (2009), 713–721.

[17] S.K. Sardar, R. Mukherjee, On additively regular seminear-rings, Semigroup Forum, **88,** no. 3, (2014), 541–554.

[18] H.S. Vandiver, Note on a simple type of algebra in which the cancellation law of addition does not hold, Bull. Amer. Math. Soc., **40,** (1934), 914–920.