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Very Short Note on Burning Trees with Sufficiently Long Arms

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Abstract

Burning number is a discrete graph algorithm parameter inspired by the spread of social contagion. Bonato et al. conjectured in 2016 that for any connected graph of order N^2, the burning number is at most N. In this work, we prove that an earlier slightly strengthened burning number conjecture holds for trees with sufficiently long arms.

1 Introduction

In 2016, Bonato et al. introduced graph burning as an ideal simplified model motivated to measure how fast can an influence spread throughout a social network [2]. The burning process begins with all vertices of a simple graph in an unburned state. In each round, a vertex is selected as a new burning source and vertices adjacent to an earlier burned vertices will also become burned. The *burning number* of a graph G, denoted $b(G)$, is the least number of rounds needed for all vertices of the graph to be burned. We say that G is *m-burnable* if G can be completely burned in m rounds.

Studies of graph burning have focused on trees because the burning number for any connected graph is the minimum burning number of its spanning

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trees [2]. It was conjectured early on that T is m-burnable for any tree T with m^2 vertices [2]. Recently, a stronger conjecture, which says that every tree with n leaves of order $m^2 + n - 2$ is m-burnable with exceptions when $m \leq n$, was established to hold for spiders [3] and double spiders [4]. In this very short exposition, we aim to provide extra support to the stronger burning number conjecture by verifying it for all trees with sufficiently long arms. For basic facts and more literature review and references, the reader can refer to a comprehensive survey [1].

Finally, standard terminology and notations in graph theory are mostly adopted and assumed to be understood. Every leaf of a tree is connected to the closest branch vertex by a unique path, called an *arm*, whereas the path connecting two branch vertices are called an *internal path*. The *first vertex* of an arm refers to the vertex on the arm next to the branch vertex. The set of all vertices belonging to an internal path including the branch vertices is denoted by body(T).

2 Main results

For $n \geq 2$, let L_n denote the least integer with the property that: if T is a path forest of order m^2 with n paths each of order at least L_n, then $b(T) = m$. The existence of L_n was proved in [4]. Our first theorem implies that any tree of order $m^2 + n - 2$ with n arms each of length at least $L_n + m - 1$ is m-burnable. Our second theorem improves the sufficient length but only for the case of trees with three branch vertices.

Lemma 2.1. *Let $m \geq 2$. Suppose T is a tree such that $|V(T)| \leq m^2+m-1$ and all arms have length $m-1$. There exist $x_1, x_2, \ldots, x_t \in V(T)$ such that*

1. $\{ x_i \mid 1 \leq i \leq t \} \subseteq \mathrm{body}(T) \subseteq \bigcup_{i=1}^{t} N_{m-i}[x_i]$*; and*
2. $\left|\bigcup_{i=1}^{t} N_{m-i}[x_i]\right| \geq \sum_{i=1}^{t} [2m - (2i-1)] + m - 1.$

Proof. We argue by induction on the number of branch vertices. The base case is trivial as there is only one branch vertex. For the induction step, suppose T has k branch vertices with $k \geq 2$ and $|V(T)| \leq m^2+m-1$. Select a branch vertex v such that there is a unique internal path P that joins v to another branch vertex, say u. Next, remove the internal path including v and all arms joined to v, but join a new arm (with new vertices) of length $m-1$ to u, resulting in a tree T'. Applying the induction hypothesis on T', we obtain $x_1, x_2, \ldots, x_t \in \mathrm{body}(T') \subseteq \mathrm{body}(T)$. If $\mathrm{body}(T) \subseteq \bigcup_{i=1}^{t} N_{m-i}^{T}[x_i]$, then by the definition of T', it is easy to see that $\left|\bigcup_{i=1}^{t} N_{m-i}^{T}[x_i]\right| \geq \left|\bigcup_{i=1}^{t} N_{m-i}^{T'}[x_i]\right|$ and thus $x_1, x_2, \ldots, x_t$ have the required property.

Now, suppose $E = \text{body}(T) \setminus \bigcup_{i=1}^{t} N_{m-i}^{T}[x_i] \neq \emptyset$. Note that E is a segment of the internal path P between v and a vertex in P and $|\bigcup_{i=1}^{t} N_{m-i}^{T}[x_i]| = |\bigcup_{i=1}^{t} N_{m-i}^{T'}[x_i]|$. Clearly, there is a least t' such that $E \subseteq \bigcup_{i=t+1}^{t+t'} N_{m-i}^{T}[x_i]$ and $\left|\bigcup_{i=t+1}^{t+t'} N_{m-i}^{T}[x_i]\right| \geq \sum_{i=t+1}^{t+t'} [2m-(2i-1)]$ for some $x_{t+1}, x_{t+2}, \ldots, x_{t+t'} \in E$ where $N_{m-i}^{T}[x_i]$ for $t+1 \leq i \leq t+t'-1$ induce disjoint subpaths of E and $x_{t+t'}$ can be chosen to be v when necessary. (The condition $|V(T)| \leq m^2+m-1$ conveniently ensures that t' exists.) Therefore, $x_1, x_2, \ldots, x_{t+t'}$ have the required property. □

Theorem 2.2. *Let $m > n \geq 3$. Suppose T is a tree with n arms each of length at least $L_n + m - 1$. If $|V(T)| \leq m^2 + m - 1$, then T is m-burnable.*

Proof. Consider the subtree T' of T by reducing the length of every arm to $m-1$. Applying Lemma 2.1 on T', it follows that for some t, all vertices in body(T) can be burned in m rounds using the first t burning sources placed within body(T). The fire from each of the t burning sources can burn at most $m-1$ vertices on each arm. The remaining unburned vertices form a path forest of order at most $m^2+m-1-(\sum_{i=1}^{t}[2m-(2i-1)]+m-1) = (m-t)^2$ vertices, where each path has order at least L_n. Therefore, it can be burned by the remaining $m-t$ burning sources by the definition of L_n. □

For the remainder of this section, let $m > n \geq 5$. Suppose T is a tree with three branch vertices and n arms. Let v_l^{br}, v_r^{br}, and v_m^{br} denote the left, right, and middle branch vertices of T, respectively. Also, let $d = |\,\text{body}(T)|$.

Lemma 2.3. *Suppose all arms of T have length L with $3 \leq 4m-2-4L \leq d \leq 4m-4-2L$. Let $r = 4m-4-2L-d$. Then there exist $v_1, v_2 \in \text{body}(T)$ such that $N = N_{m-1}[v_1] \cup N_{m-2}[v_2]$ contains every vertex of T except possibly the last $L - \lfloor \frac{r}{2} \rfloor - 1$ vertices of any arm joined to v_m^{br} and thus $|N| \geq 4m+n-6$.*

Proof. Let $v_1, v_2 \in \text{body}(T)$ be the vertices given by $\text{dist}(v_1, v_l^{br}) = m-1-L$ and $\text{dist}(v_2, v_r^{br}) = m-2-L$. If $\text{dist}(v_1, v_m^{br}) \leq m - \lfloor \frac{r}{2} \rfloor - 2$ or $\text{dist}(v_2, v_m^{br}) \leq m - \lfloor \frac{r}{2} \rfloor - 3$, then v_1 and v_2 has the required property. It can be verified that v_m^{br} satisfies the disjunctive property unless v_m^{br} is one of at most two vertices (labelled by $*$ in Figure 1). In the exceptional case(s), it suffices to swap the choices of v_1 and v_2. Since at least two arms are joined to each v_l^{br} and v_r^{br}, it follows that $|N| \geq d + 4L + (n-4)\left(\lfloor \frac{r}{2} \rfloor + 1\right) \geq (4m-2) + (n-4)$. □

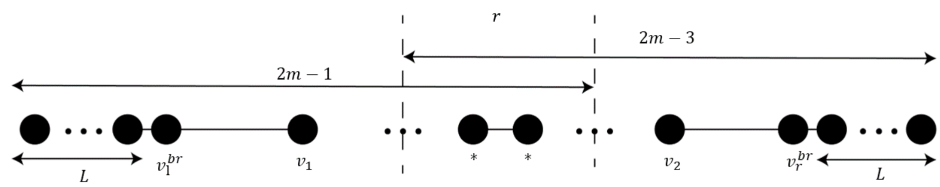

Figure 1: The two exceptional cases for the case r is even.

Lemma 2.4. *Suppose all arms of T have length $L = \lceil \frac{2m-4}{n} \rceil + 1$. If $2m-2 \leq d \leq 4m-6$, then there exist $v_1, v_2 \in \operatorname{body}(T)$ such that $N = N_{m-1}[v_1] \cup N_{m-2}[v_2] \supseteq \operatorname{body}(T)$ and $|N| \geq 4m+n-6$.*

Proof. For the case $4m-4-2L \leq d \leq 4m-6$, first we let $l_1 = \lfloor \frac{4m-4-d}{2} \rfloor$ and $l_2 = \lceil \frac{4m-4-d}{2} \rceil$. Note that $l_1 + l_2 + d = 4m-4$. Clearly, we can choose $v_1, v_2 \in \operatorname{body}(T)$ such that N includes $\operatorname{body}(T)$ and contains the first l_1 vertices of any arm joined to v_l^{br} and the first l_2 vertices of any arm joined to v_r^{br}. By swapping v_1 and v_2 if necessary, N contains at least the first vertex of any arm joined to v_m^{br}. Hence,

$$|N| \geq 2(l_1+l_2)+n-4+d = 8m+n-12-d \geq 8m+n-12-4m+6 = 4m+n-6.$$

The case $4m-2-4L \leq d \leq 4m-4-2L$ is taken care by Lemma 2.3. In particular, when $d = 4m-2-4L$, there exist $v_1, v_2 \in \operatorname{body}(T)$ such that N contains all vertices of T. It follows that such v_1 and v_2 exist when $2m-2 \leq d \leq 4m-2-4L$. It remains to note that when $d = 2m-2$, $|V(T)| = d + nL \geq 2m-2+n\left(\frac{2m-4}{n}+1\right) = 4m+n-6$. □

Theorem 2.5. *Suppose every arm of T has length at least $L_n + \lceil \frac{2m-4}{n} \rceil + 1$. If the order of T is at most m^2+n-2, then T is m-burnable.*

Proof. Let t be the least such that $d+2 \leq \sum_{i=1}^{t}[2m-(2i-1)]$. First, consider the case $t = 2$. Applying Lemma 2.4 on the subtree of T with the length of every arm reduced to $\lceil \frac{2m-4}{n} \rceil + 1$, it follows that the first two burning sources can be chosen such that it burns at least $4m+n-6$ vertices of T in m rounds such that the remaining unburned vertices form a path forest of order at most $(m-2)^2$ consisting of n paths each of order at least L_n. Hence, the path forest can be burned using the remaining $m-2$ burning sources.

Now, suppose $t > 2$. Let d_1 (respectively, d_2) denote the number of vertices strictly between v_m^{br} and v_l^{br} (respectively, v_r^{br}). Note that $\max\{d_1, d_2\} \geq 2m-4$ because $d > 4m-6$. Hence, we can find I_1 and I_2 such that

$I_1 \cup I_2 = \{3, 4, \ldots, t\}$, $\sum_{i\in I_1}[2m-(2i-1)] \leq d_1$, and $\sum_{i\in I_2}[2m-(2i-1)] \leq d_2$. Furthermore, if both I_1 and I_2 are nonempty, we specially require that $3 \in I_1$ and $4 \in I_2$. Let T' be obtained from T by shortening the two internal paths of T by $\sum_{i\in I_1}[2m-(2i-1)]$ and $\sum_{i\in I_2}[2m-(2i-1)]$ respectively and reducing the length of every arm to $\lceil \frac{2m-4}{n} \rceil + 1$. The conditions on I_1 and I_2 would ensure that the next $t-2$ burning sources, alongside with the first two burning sources associated correctly to the two vertices obtained by applying Lemma 2.4 on T', can be chosen such that the remaining unburned vertices of T in m rounds form a path forest of order at most $(m-t)^2$ with n paths each of order at least L_n. (The special requirement ensures that the additional $t-2$ burning sources all stay within body(T).)

The proof for the case $t = 1$ is much simpler and thus omitted. □

As a conclusion, we conjecture that Theorem 2.5 holds generally for any tree (with any number of branch vertices).

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