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On the diameter of generalized Petersen graphs

Laila Loudiki, Mustapha Kchikech

Department of Mathematics and Computer Science
Polydisciplinary Faculty of Safi
Cadi Ayyad University
B.P. 4162, Sidi Bouzid
Safi 46000, Morocco

email: laila.loudiki@ced.uca.ma, m.kchikech@uca.ac.ma

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Abstract

When we contract the edges of the generalized Petersen graph $GPG(n, s)$, we end up with the widely recognized circulant graph $C_n(1, s)$. Conversely, we can construct $GPG(n, s)$ from $C_n(1, s)$. This natural connection between the two types of graphs led Beenker and Van Lint [8] to demonstrate that if $C_n(1, s)$ has a diameter of d, then $GPG(n, s)$ will have a diameter that is at least $d + 1$ and at most $d+2$. In this paper, we outline the necessary and sufficient conditions for the diameter of $GPG(n, s)$ to be equal to $d+1$, as well as providing sufficient conditions for it to equal $d + 2$. Moreover, we introduce an algorithm capable of computing the diameter of a generalized Petersen graph in a time complexity of $\mathcal{O}(\log n)$.

1 Introduction

For integers n and s with $n \geq 5$, the *generalized Petersen graph* $GPG(n, s)$ was defined in [1] (a subclass with n and s relatively prime considered already

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by Coxeter [2]) to have vertex-set $V(GPG(n,s)) = \{u_i, v_i : i \in \mathbb{Z}_n\}$. The edge-set may be naturally partitioned into three equal parts (note that all subscripts are assumed modulo n): the *outer edges* $E_{\mathsf{O}}(n,s) = \{u_i u_{i+1} : i \in \mathbb{Z}_n\}$, the *inner edges* $E_{\mathsf{I}}(n,s) = \{v_i v_{i+s} : i \in \mathbb{Z}_n\}$, and the *spokes* $E_{\mathsf{S}}(n,s) = \{u_i v_i : i \in \mathbb{Z}_n\}$. Thus the edge-set may be defined as $E(GPG(n,s)) = E_{\mathsf{O}}(n,s) \cup E_{\mathsf{I}}(n,s) \cup E_{\mathsf{S}}(n,s)$. We call the u_i vertices the *outer vertices* and the v_i vertices the *inner vertices*.

The distance $d_p(i,j)$ between two vertices i and j in $GPG(n,s)$ is the length of a shortest path joining i and j. The diameter of $GPG(n,s)$, denoted by $diam(GPG(n,s))$, is the maximum distance among all pairs of vertices in $GPG(n,s)$. Here we are interested in the following problem.

Problem. Given n, s, determine $diam(GPG(n,s))$.

The exact calculation of the diameter of generalized Petersen graphs is a well-studied problem: Krishnamoorthy and Krishnamurthy [3] proved that the diameter of $GPG(n,2)$ is $O(\frac{n}{4})$ when n is odd. Xinmin and Tianming [4] showed that the diameter of $GPG(n,2)$ equals the same previous value when n is even. Zhang et al. [5] proved that the diameter of $GPG(n,s)$ is $O(\frac{n}{2s})$ where $s \geq 3$. Ekinci and Gauci [6] have proved that the diameter of $GPG(ts,s)$ is $\lfloor \frac{t+s+3}{2} \rfloor$ for $t \geq 3$ and $s \geq 2$. However, there were no formulas giving exact values for the diameter of $GPG(n,s)$ for all n and s.

2 Circulant graphs

*Circulant graphs*form an interesting and well-studied class of graphs [7]: Given $n \geq 4$ and $2 \leq s \leq \lfloor \frac{n-1}{2} \rfloor$, the circulant graph $C_n(1,s)$ has $\mathbb{Z}_n$ as a vertex set and in which two distinct vertices i and j are adjacent if and only if $|i-j|_n \in \{1, s\}$, where $|x|_n = \min(|x|, n-|x|)$ is the circular distance modulo n.

Definition 2.1. *In $C_n(1,s)$, a path from a vertex i to another vertex j is denoted as $P_c(i,j)$ and is represented as the pair $(\alpha a^{\pm}, \beta c^{\pm})$, where a (or c) indicates that $P_c(i,j)$ traverses outer (or inner) edges; α (or β) denotes the number of outer (or inner) edges; The symbol $+$ (or $-$) indicates that $P_c(i,j)$ follows the clockwise (or counterclockwise) direction.*

Notation 2.2. *Let i and j be two vertices of $C_n(1,s)$.*

An outer (or inner) edge connecting the vertices i and j and following the clockwise $(+)$ or counterclockwise $(-)$ direction is represented as $i \rightsquigarrow^{a^{\pm}} j$ (or $i \rightsquigarrow^{c^{\pm}} j$), respectively.

The length of the path $P_c(i,j)$ is denoted by $\ell(P_c(i,j))$, the number of outer edges in $P_c(i,j)$ is represented by $\ell_a(P_c(i,j))$, and the number of inner edges in $P_c(i,j)$ is denoted by $\ell_c(P_c(i,j))$.

Circulant graphs can be obtained from generalized Petersen graphs by *contracting* the spokes $E_{\mathsf{S}}(n,s)$. By a reversed procedure generalized Petersen graphs can be obtained from circulant graphs (see Figure 1).

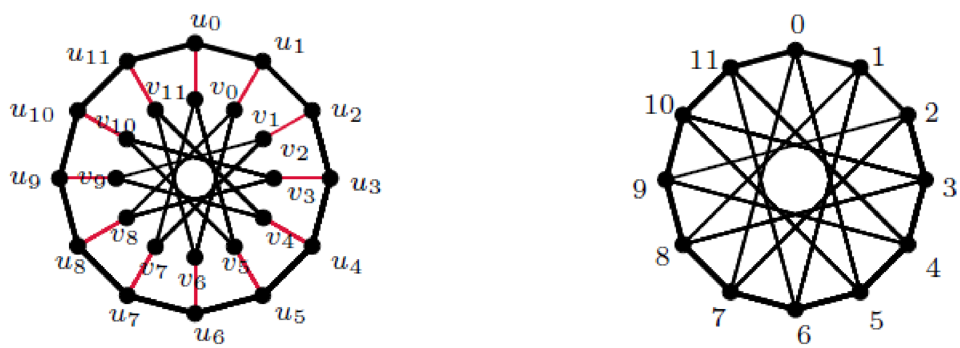

Figure 1: $GPG(12,5)$ and $C_{12}(1,5)$.

Notation 2.3. *Let $i \in V(C_n(1,s))$ and $u_i, v_i \in V(GPG(n,s))$ such that $(u_i,v_i) \in E(GPG(n,s))$.*

Let $\mathcal{T}_{\mathsf{g}}$ be the transformation of $GPG(n,s)$ into $C_n(1,s)$. $\mathcal{T}_{\mathsf{g}}$ is based on contracting the spokes from $GPG(n,s)$.

We combine the vertices u_i and v_i into one single vertex denoted $w_i = \{u_i, v_i\}$. The notation $i \equiv w_i$ means that after applying $\mathcal{T}_{\mathsf{g}}$, the vertex i is the equivalent of w_i (in terms of labeling).

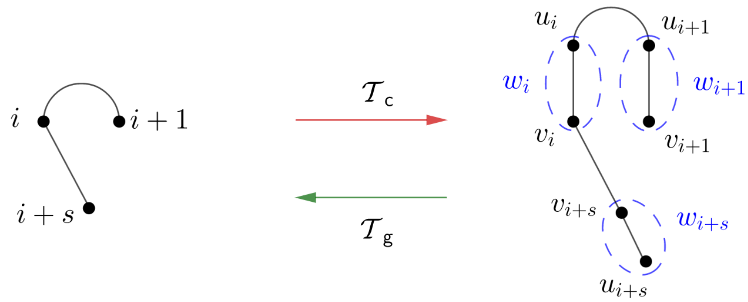

Figure 2: $\mathcal{T}_{\mathsf{g}}$ and $\mathcal{T}_{\mathsf{c}}$.

Definition 2.4. *We define the transformation of $C_n(1,s)$ into $GPG(n,s)$, denoted by $\mathcal{T}_{\mathsf{c}}$, as follows.*
$i \in V(C_n(1,s)) \Rightarrow w_i \in V(GPG(n,s))$ *such that* $i \equiv w_i$.
$(i, i+s) \in E(C_n(1,s)) \Rightarrow (u_i, v_i), (v_i, v_{i+s}), (v_{i+s}, u_{i+s}) \in E(GPG(n,s))$.
$(i, i+1) \in E(C_n(1,s)) \Rightarrow (u_i, u_{i+1}), (u_i, v_i), (u_{i+1}, v_{i+1}) \in E(GPG(n,s))$, $(v_i, v_{i+1}) \notin E(GPG(n,s))$.

Notation 2.5. *In $GPG(n,s)$, we denote a path leading from a vertex x to another vertex y by $P_p(x,y)$. We denote the length of $P_p(x,y)$ by $\ell(P_p(x,y))$, the number of outer edges of $P_p(x,y)$ by $\ell_a(P_p(x,y))$, the number of inner edges of $P_p(x,y)$ by $\ell_c(P_p(x,y))$, and the number of spokes of $P_p(x,y)$ by $\ell_s(P_p(x,y))$.*

We represent an outer (resp. inner) edge connecting two vertices x and y of $GPG(n,s)$ and taking the clockwise $(+)$ or the counterclockwise $(-)$ direction by $x \rightsquigarrow^{a^{\pm}} y$ (resp. $x \rightsquigarrow^{c^{\pm}} y$). A spoke connecting the vertices x and y can be represented by $x \rightsquigarrow^{spoke} y$.

3 Diameter of generalized Petersen graphs

The following result shows the distance between any pair of vertices in the generalized Petersen graph.

Theorem 3.1. *Let $i, j \in V(C_n(1,s))$ and $w_i, w_j \in V(GPG(n,s))$ such that $i \equiv w_i$ and $j \equiv w_j$. Let x_i and y_j be two vertices of $GPG(n,s)$ such that $x_i \in w_i$ and $y_j \in w_j$. We have*

$$d_c(i,j) \leq d_p(x_i, y_j) \leq d_c(i,j) + 2.$$

Proof. Let i and j be two arbitrary vertices in $C_n(1,s)$. Let $w_i, w_j, x_i, y_j \in V(GPG(n,s))$ such that $i \equiv w_i$, $j \equiv w_j$, $x_i \in w_i$ and $y_j \in w_j$.

Let $P_{sg}(x_i, y_j)$ denote the shortest path between x_i and y_j in $V(GPG(n,s))$. Let $d_p(x_i, y_j) = \ell_a(P_{sg}(x_i, y_j)) + \ell_c(P_{sg}(x_i, y_j)) + \ell_s(P_{sg}(x_i, y_j)) = \ell_a + \ell_c + \ell_s = \ell$ where $\ell_s \geq 0$. After applying $\mathcal{T}_{\mathsf{g}}$, there exists a path $P_c(i,j)$ in $C_n(1,s)$ of length $\ell(P_c(i,j)) = \ell - \ell_s \leq \ell$. Thus, $d_c(i,j) \leq \ell(P_c(i,j)) \leq \ell$. Therefore, $d_c(i,j) \leq d_p(x_i, y_j)$.

Let $P_{sc}(i,j)$ denote the shortest path between i and j in $C_n(1,s)$. Assume that $d_c(i,j) = \ell(P_{sc}(i,j)) = \ell'$. Since $P_{sc}(i,j)$ walks through all the outer edges before entering to the inner edges, after applying $\mathcal{T}_{\mathsf{c}}$, there exists a path $P_p(x_i, y_j)$ in $GPG(n,s)$ that also walks through all the outer edges before entering to the inner edges and leading from x_i to y_j. Thus, depending on whether x_i and y_j are inner or outer vertices, $P_p(x_i, y_j)$ will contain at most two spokes*; i.e., $\ell(P_p(x_i, y_j)) \leq \ell' + 2$. Therefore, $d_p(x_i, y_j) \leq d_c(i, j) + 2$. $\square$

Since any undirected circulant graph is vertex-transitive, for any pair of vertices i and j within $C_n(S)$, their distance, denoted as $d_c(i, j)$, can be equivalently expressed as $d_c(0, z)$, where $z = \begin{cases} j - i, & \text{if } i < j, \\ n - i + j, & \text{otherwise.} \end{cases}$
Henceforth, we denote paths leading from the vertex 0 to another vertex i in $C_n(1, s)$ as $P_c(i)$. The distance of $P_c(i)$ is represented as $d_c(i)$.

Beenker and Van Lint [8] have proved that if $C_n(1, s)$ has diameter d, then $GPG(n, s)$ has diameter at least $d + 1$ and at most $d + 2$. It is easy to verify that for $n \in \{5, 6, 7\}$, $diam(GPG(n, s)) = diam(C_n(1, s)) + 1$. For $n \geq 8$, we present the following result.

Theorem 3.2. *Let $n \geq 8$ and $2 \leq s \leq \lfloor \frac{n-1}{2} \rfloor$ and let $V_{diam(C_n(1,s))} = \{i \in V(C_n(1, s)) : d_c(i) = diam(C_n(1, s))\}$. $diam(GPG(n, s)) = diam(C_n(1, s)) + 1$ if and only if the following assessments are satisfied:*

1. *For all vertices i in $V_{diam(C_n(1,s))}$, there exists a path $P_c(i)$ walking only through outer edges such that $\ell(P_c(i)) = diam(C_n(1, s))$;*

2. *For all vertices i in $V_{diam(C_n(1,s))}$, there exists a path $P'_c(i)$ walking only through inner edges such that $\ell(P'_c(i)) = diam(C_n(1, s))$.*

Proof. Assume that $diam(GPG(n, s)) = diam(C_n(1, s)) + 1 = d + 1$. Let $i \in V_{diam(C_n(1,s))}$ and let $x_0, y_i \in V(GPG(n, s))$ such that $0 \equiv w_0$, $i \equiv w_i$, $x_0 \in w_0$ and $y_i \in w_i$.

Case 1. $x_0 = u_0$

Let $P_p(x_0, y_i)$ be a path in $GPG(n, s)$ of length $d + 1$ represented as follows. $u_0 \rightsquigarrow^{a^+} u_1 \rightsquigarrow^{a^+} \ldots \rightsquigarrow^{a^+} u_i \rightsquigarrow^{\text{spoke}} v_i$ (or $u_0 \rightsquigarrow^{a^-} u_{n-1} \rightsquigarrow^{a^-} \ldots \rightsquigarrow^{a^-} u_i \rightsquigarrow^{\text{spoke}} v_i$). We have $\ell(P_p(u_0, u_i)) = d$. So, $d_p(u_0, u_i) \leq \ell(P_p(u_0, u_i)) \leq d$. However, by Theorem 3.1, $d_p(u_0, u_i) \geq d$. Thus, $d_p(u_0, u_i) = d$. Moreover, $\ell(P_p(u_0, v_i)) = d + 1$. So $d_p(u_0, v_i) \leq \ell(P_p(u_0, v_i)) \leq d + 1$. If we assume that $d_p(u_0, v_i) = d$, then there exists a shortest path $P_{sp}(u_0, v_i)$ in $GPG(n, s)$ such that $d_p(u_0, v_i) = \ell(P_{sp}(u_0, v_i)) = d$.

Case 1.1. $P_{sp}(u_0, v_i)$ walks only through outer edges

Since $P_{sp}(u_0, v_i)$ is represented by $u_0 \rightsquigarrow^{a^+} u_1 \rightsquigarrow^{a^+} \ldots \rightsquigarrow^{a^+} u_i \rightsquigarrow^{\text{spoke}} v_i$ (or by $u_0 \rightsquigarrow^{a^-} u_{n-1} \rightsquigarrow^{a^-} \ldots \rightsquigarrow^{a^-} u_i \rightsquigarrow^{\text{spoke}} v_i$), after applying $\mathcal{T}_{\mathsf{g}}$, there exists a path $P_c(i)$ in $C_n(1, s)$ of length equals to $d - 1$. However, since $i \in V_{diam(C_n(1,s))}$, $d_c(i) = d$. This is a contradiction.

Case 1.2. $P_{sp}(u_0, v_i)$ walks only through inner edges

In this case, $P_{sp}(u_0, v_i)$ is represented by $u_0 \rightsquigarrow^{\text{spoke}} v_0 \rightsquigarrow^{c^+} v_s \rightsquigarrow^{c^+} \ldots \rightsquigarrow^{c^+} v_i$ (or by $u_0 \rightsquigarrow^{\text{spoke}} v_0 \rightsquigarrow^{c^-} v_{n-s} \rightsquigarrow^{c^-} \ldots \rightsquigarrow^{c^-} v_i$), after applying $\mathcal{T}_{\mathsf{g}}$, there exists a path $P_c(i)$ in $C_n(1, s)$ of length equals to $d - 1$. This is a contradiction because $i \in V_{diam(C_n(1,s))}$.

Case 1.3. $P_{sp}(u_0, v_i)$ walks through inner and outer edges

In this case, $P_{sp}(u_0, v_i) = (\alpha a^{\pm}, \beta c^{\pm})$, $\alpha \geq 1$, $\beta \geq 1$, $\alpha + \beta = d$, and it is represented as follows:
$u_0 \rightsquigarrow^{a^+} u_1 \rightsquigarrow^{a^+} \ldots \rightsquigarrow^{a^+} u_\alpha \rightsquigarrow^{\text{spoke}} v_\alpha \rightsquigarrow^{c^+} v_{\alpha+s} \rightsquigarrow^{c^+} \ldots \rightsquigarrow^{c^+} v_i$ (or $u_0 \rightsquigarrow^{a^-} u_{n-1} \rightsquigarrow^{a^-} \ldots \rightsquigarrow^{a^-} u_{n-\alpha} \rightsquigarrow^{\text{spoke}} v_{n-\alpha} \rightsquigarrow^{c^-} v_{n-\alpha-s} \rightsquigarrow^{c^-} \ldots \rightsquigarrow^{c^-} v_i$), after applying $\mathcal{T}_{\mathsf{g}}$, there exists a path $P_c(i)$ in $C_n(1, s)$ of length $d-1$. This is also a contradiction.

Consequently, $d_p(u_0, v_i) = d + 1$ and $d_p(u_0, u_i) = d$. Since $P_p(u_0, y_i)$, $y_i \in w_i$, walks only through outer edges, after applying $\mathcal{T}_{\mathsf{g}}$, there exists a path $P_c(i)$ in $C_n(1, s)$ walking also through outer edges such that $\ell(P_c(i)) = d$.

Case 2. $x_0 = v_0$

Let $P'_p(x_0, y_i)$ be a path in $GPG(n, s)$ of length $d+1$ represented as follows: $v_0 \rightsquigarrow^{c^+} v_s \rightsquigarrow^{c^+} \ldots \rightsquigarrow^{c^+} v_i \rightsquigarrow^{\text{spoke}} u_i$ (or $v_0 \rightsquigarrow^{c^-} v_{n-s} \rightsquigarrow^{c^-} \ldots \rightsquigarrow^{c^-} v_i \rightsquigarrow^{\text{spoke}} u_i$). Since $\ell(P'_p(v_0, v_i)) = d$, we get $d_p(v_0, v_i) \leq \ell(P'_p(v_0, v_i)) \leq d$. However, by Theorem 3.1, $d_p(v_0, v_i) \geq d$. Thus $d_p(v_0, v_i) = d$. Moreover, $\ell(P'_p(v_0, u_i)) = d + 1$. So, $d_p(v_0, u_i) \leq \ell(P'_p(v_0, u_i)) \leq d + 1$. We proceed similarly as in the previous case in order to prove that $d_p(v_0, u_i) = d + 1$. Consequently, $d_p(v_0, v_i) = d$ and $d_p(v_0, u_i) = d + 1$. Since $P'_p(v_0, y_i)$, $y_i \in w_i$ walks only through inner edges, after applying $\mathcal{T}_{\mathsf{g}}$, there exists a path $P'_c(i)$ in $C_n(1, s)$ walking only through inner edges such that $\ell(P'_c(i)) = d$.

Let $i \in V_{diam(C_n(1,s))}$. Suppose that there exists a path $P_c(i)$ walking only through outer edges such that $\ell(P_c(i)) = diam(C_n(1, s))$, as well as another path $P'_c(i)$ walking only through inner edges such that $\ell(P'_c(i)) = diam(C_n(1, s))$. Next, we prove that $diam(GPG(n, s)) = diam(C_n(1, s)) + 1 = d + 1$. Let $x_0, y_i \in V(GPG(n, s))$ such that $0 \equiv w_0$, $i \equiv w_i$, $x_0 \in w_0$ and $y_i \in w_i$.

Case 1. $x_0 = u_0$

Since there exists a path $P_c(i)$ walking only through outer edges such that $\ell(P_c(i)) = d$, after applying $\mathcal{T}_{\mathsf{c}}$, there exists a path $P_p(x_0, y_i)$ in $GPG(n, s)$ represented as follows. $u_0 \rightsquigarrow^{a^+} u_1 \rightsquigarrow^{a^+} \ldots \rightsquigarrow^{a^+} u_i \rightsquigarrow^{\text{spoke}} v_i$ (or $u_0 \rightsquigarrow^{a^-} u_{n-1} \rightsquigarrow^{a^-} \ldots \rightsquigarrow^{a^-} u_i \rightsquigarrow^{\text{spoke}} v_i$). We have $\ell(P_p(u_0, v_i)) = d + 1$ and $\ell(P_p(u_0, u_i)) = d$. Thus $d_p(u_0, y_i) \leq \ell(P_p(u_0, y_i)) \leq d + 1$ for all $y_i \in w_i$.

Case 2. $x_0 = v_0$

Similarly, because there exists a path $P'_c(i)$ walking only through inner edges such that $\ell(P'_c(i)) = d$, after applying $\mathcal{T}_\mathsf{c}$, there exists a path $P'_p(x_0, y_i)$ in $GPG(n, s)$ represented as follows. $v_0 \rightsquigarrow^{c^+} v_s \rightsquigarrow^{c^+} \dots \rightsquigarrow^{c^+} v_i \rightsquigarrow^{\text{spoke}} u_i$ (or $v_0 \rightsquigarrow^{c^-} v_{n-s} \rightsquigarrow^{c^-} \dots \rightsquigarrow^{c^-} v_i \rightsquigarrow^{\text{spoke}} u_i$). We have $\ell(P'_p(v_0, v_i)) = d$ and $\ell(P'_p(v_0, u_i)) = d + 1$. Thus, $d_p(v_0, y_i) \leq \ell(P'_p(v_0, y_i)) \leq d + 1$ for all $y_i \in w_i$.

Therefore, $d_p(x_0, y_i) \leq d + 1$ for all $x_0, y_i \in V(GPG(n, s))$. Thus $diam(GPG(n, s)) \leq d+1$. Furthermore, by [8], we have $diam(GPG(n, s)) \geq d + 1$. Consequently, $diam(GPG(n, s)) = d + 1$. □

Theorem 3.3. *Let $n \geq 8$ and $2 \leq s \leq \lfloor \frac{n-1}{2} \rfloor$ and let $V_{diam(C_n(1,s))} = \{i \in V(C_n(1, s)) : \ d_c(i) = diam(C_n(1, s))\}$. If*

- *there exists $i \in V_{diam(C_n(1,s))}$ such that the shortest path between 0 and i walks either through outer edges or through inner edges;*
- *or for all $i \in V_{diam(C_n(1,s))}$, the shortest path between 0 and i walks through inner and outer edges;*
- *or there exists $i \in V_{diam(C_n(1,s))}$ such that $s \leq i \leq n - s$,*

then $diam(GPG(n, s)) = diam(C_n(1, s)) + 2$.

Proof. Assume that $diam(C_n(1, s)) = d$. By [8], $d+1 \leq diam(GPG(n, s)) \leq d + 2$. If $diam(GPG(n, s)) = d + 1$, then by Theorem 3.2, for all $i \in V_{diam(C_n(1,s))}$ there exists two paths $P_c(i)$ and $P'_c(i)$ walking respectively through outer and inner edges such that $\ell(P_c(i)) = \ell(P'_c(i)) = d$. This contradicts the theorem's first two conditions. Thus $diam(GPG(n, s)) = d + 2$.

When a vertex $i \in V_{diam(C_n(1,s))}$ is located between s and $n - s$, it is preferable to take a path containing the inner edges than choose a path walking only through outer edges. Thus, for all $i \in V_{diam(C_n(1,s))}$ such that $s \leq i \leq n - s$, the shortest path between 0 and i in $C_n(1, s)$ will not walk only by outer edges. Therefore, by Theorem 3.2, $diam(GPG(n, s)) \neq d + 1$. Consequently, by [8], $diam(GPG(n, s)) = d + 2$. □

Conjecture 3.4. *For all n and s,*

$$diam(GPG(n, s)) = \begin{cases} diam(C_n(1, s)) + 1 & \text{if } n = 4p \text{ and } s = 2p - 1, p > 2, \\ diam(C_n(1, s)) + 2 & \text{otherwise.} \end{cases}$$

4 Algorithm for the diameter of generalized Petersen graphs

In [9], Zerovnik and Pisanski proposed a method for computing the diameter of circulant graphs $C_n(s_1, s_2)$ with a running time of $\mathcal{O}(\log n)$. In particular, there is an algorithm that computes the diameter of a circulant network $C_n(1, s)$ in $\mathcal{O}(\log n)$ time. This algorithmic contribution does not give an exact value for the diameter of circulant graphs. As $diam(GPG(n, s)) = diam(C_n(1, s)) + \varepsilon$, $\varepsilon \in \{1, 2\}$, we get the following result;

Theorem 4.1. *There is an algorithm that computes the diameter of* $GPG(n, s)$ *with running time* $\mathcal{O}(\log n)$.

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