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On the 2-normed Orlicz space

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Abstract

We study the Orlicz space $L_\Phi(X)$ that is equipped with a 2-norm. This can be viewed as a generalization of the usual norm in the Orlicz space $L_\Phi(X)$. Using a derived norm that is obtained from the 2-norm, we show that $L_\Phi(X)$ is complete with respect to the 2-norm.

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1 Introduction

The theory of 2-normed spaces was initially introduced by Gähler [1] in the mid 1960's and its generalization can be found in [2, 3, 4]. Since then, many researchers have studied the structures of these spaces. Recent results can be found, for example, in [5, 6, 7, 9, 8]. In 2013, Idris et al. [10] studied the 2-normed spaces on the space of p-summable sequences. Ekariani et al. [5] also studied the 2-normed space of p-integrable functions on the Lebesgue spaces. Before we present our main results, here are some basic concepts of the 2-normed spaces.

Let X be a real vector space of dimension d where $2 \leq d < \infty$. A 2-norm is a mapping $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following four conditions:

1. $\|x, y\| = 0$ if and only if x, y are linearly dependent;
2. $\|x, y\| = \|y, x\|$ for every $x, y \in X$;
3. $\|\alpha x, y\| = |\alpha| \, \|x, y\|$ for every $x, y \in X$ and for every $\alpha \in \mathbb{R}$;
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for every $x, y, z \in X$.

The pair $(X, \|\cdot,\cdot\|)$ is called a 2-*normed space*. Using this definition, we have $\|x, y\| \geq 0$ and $\|x, y\| = \|x, y + \alpha x\|$ for any $x, y \in X$ and $\alpha \in \mathbb{R}$. A sequence $\{x_n\}$ in the 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be *convergent* to an x in X if $\lim\limits_{n\to\infty} \|x_n - x, y\| = 0$ for any $y \in X$. A sequence $\{x_n\}$ is said to be a *Cauchy sequence* in X if for any $y \in X$ and $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|x_n - x_m, y\| < \epsilon$ for any $n, m \geq n_0$. If every Cauchy sequence converges to an x in X, then X is said to be *complete*. Any complete 2-normed space is said to be a 2-Banach space.

Let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function (that is, Φ is convex, left-continuous, $\Phi(0) = 0$, and $\lim\limits_{t\to\infty} \Phi(t) = \infty$). *The Orlicz space* $L_\Phi(X)$ is defined as the set of measurable functions $f : X \to \mathbb{R}$ such that $\int_X \Phi(a|f(x)|)dx < \infty$ for some $a > 0$. The Orlicz space $L_\Phi(X)$ is a Banach space with respect to the usual norm:

$$\|f\|_{L_\Phi} := \inf \left\{ b > 0 : \int_X \Phi\left(\frac{|f(x)|}{b}\right) dx \leq 1 \right\}$$

(see [11, 12, 13]). Note that, if if $\Phi(t) := t^p$ for some $p \geq 1$, then $L_\Phi(X) = L_p(X)$, the Lebesgue space of p-th integrable functions on X. Thus, the Orlicz space $L_\Phi(X)$ can be viewed as a generalization of the Lebesgue space $L^p(X)$.

In this paper, we introduce the Orlicz space $L_\Phi(X)$ equipped with the 2-norm, which can be regarded as a generalization of the usual norm. Moreover, we define a norm that is obtained from the 2-norm and show that $L_\Phi(X)$ is a 2-Banach space with respect to its 2-norm.

2 Main results

2.1 $L_\Phi(X)$ as a 2-normed space

Let $L_\Phi(X)$ be the Orlicz space where $\Phi : [0,\infty) \to [0,\infty)$ is the Young function and X is a measure space with at least n disjoint subsets of positive measure. We define the mapping $\|\cdot,\cdot\|_{L_\Phi(X)}$ on $L_\Phi(X) \times L_\Phi(X)$ by

$$\|f,g\|_{L_\Phi} := \inf\left\{ b>0 : \frac{1}{2}\int\limits_X\int\limits_X \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right) dx_1dx_2 \le 1 \right\}.$$

Next, we will show that the mapping in (2.1) defines a 2-norm on $L_\Phi(X)$. To do so, we use the following lemmas:

Lemma 2.1. *If* $0 \le \|f,g\|_{L_\Phi} < \infty$, *then*

$$\frac{1}{2}\int\limits_X\int\limits_X \Phi\left(\frac{1}{\|f,g\|_{L_\Phi}}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right) dx_1dx_2 \le 1.$$

Proof. Suppose that

$$\|f,g\|_{L_\Phi} = \inf\left\{ b>0 : \frac{1}{2}\int\limits_X\int\limits_X \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right) dx_1dx_2 \le 1 \right\}.$$

Then $\|f,g\|_{L_\Phi} = \inf B$. For any $\epsilon > 0$, there exists $b_\epsilon \in B$ such that $\|f,g\|_{L_\Phi} \le b_\epsilon \le \|f,g\|_{L_\Phi} + \epsilon$. As a consequence, we have

$$\frac{\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|}{\|f,g\|_{L_\Phi}+\epsilon} \le \frac{\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|}{b_\epsilon}.$$

for every $x_1, x_2 \in X$. By using the properties of the Young function, we have $\frac{1}{2}\int\limits_X\int\limits_X \Phi\left(\frac{1}{\|f,g\|_{L_\Phi}+\epsilon}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right) dx_1dx_2 \le 1.$

Since $\epsilon > 0$ is arbitrary, we have

$$\frac{1}{2}\int_X\int_X \Phi\left(\frac{1}{\|f,g\|_{L_\Phi}}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right)dx_1dx_2 \leq 1.$$

$\square$

Lemma 2.2. $\|f,g\|_{L_\Phi} = 0$ *if and only if* $\frac{1}{2}\int_X\int_X \Phi\left(\frac{1}{\epsilon}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right)dx_1dx_2 \leq 1$ *for every* $\epsilon > 0$.

Proof. ($\Leftarrow$) This is obvious.
($\Rightarrow$) Suppose, on the contrary, that there is $\epsilon_0 > 0$ such that

$$\frac{1}{2}\int_X\int_X \Phi\left(\frac{1}{\epsilon_0}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right)dx_1dx_2 > 1.$$

Next,

$$\|f,g\|_{L_\Phi} = \inf\left\{b > 0 : \frac{1}{2}\int_X\int_X \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right)dx_1dx_2 \leq 1\right\}.$$

Hence, $\|f,g\|_{L_\Phi} = \inf B$. Taking an arbitrary $b \in B$, we have $\epsilon_0 \neq b$. We consider two cases.

Case I: $\epsilon_0 > b$. By using the properties of the Young function, we obtain

$$\begin{aligned}&\frac{1}{2}\int_X\int_X \Phi\left(\frac{1}{\epsilon_0}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right)dx_1dx_2 \\ &\leq \frac{1}{2}\int_X\int_X \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right)dx_1dx_2 \leq 1.\end{aligned}$$

Case II: $b > \epsilon_0$. This implies that $\|f,g\|_{L_\Phi} > \epsilon_0 > 0$. Hence, we obtain a contradiction in both cases. $\square$

Lemma 2.3. $\|f,g\|_{L_\Phi} = 0$ *if and only if for every* $\alpha > 0$

$$\frac{1}{2}\int_X\int_X \Phi\left(\alpha\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right)dx_1dx_2 = 0.$$

Proof. For every $0 < \epsilon < 1$ and $\alpha > 0$, we obtain

$$\Phi\left(\alpha\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right) = \Phi\left((1-\epsilon)0 + \epsilon\left(\frac{\alpha}{\epsilon}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right)\right)$$
$$\leq \epsilon\Phi\left(\frac{\alpha}{\epsilon}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right).$$

By Lemma 2.2, we have $\frac{1}{2}\int\limits_X\int\limits_X \Phi\left(\alpha\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right) dx_1dx_2 \leq \epsilon$. Since $0 < \epsilon < 1$ is arbitrary, we conclude that

$$\frac{1}{2}\int\limits_X\int\limits_X \Phi\left(\alpha\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right) dx_1dx_2 = 0$$

for every $\alpha > 0$. Conversely, suppose that for every $\alpha > 0$

$$\frac{1}{2}\int\limits_X\int\limits_X \Phi\left(\alpha\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right) dx_1dx_2 = 0.$$

Then $\frac{1}{\alpha} \in \left\{b > 0 : \frac{1}{2}\int\limits_X\int\limits_X \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right) dx_1dx_2 \leq 1\right\}$. Hence, $\|f,g\|_{L_\Phi(X)} \leq \frac{1}{\alpha}$. Since $\alpha > 0$ is arbitrary, we have $\|f,g\|_{L_\Phi} = 0$. □

Finally, we have a 2-norm on L_Φ in the following theorem.

Theorem 2.4. *The mapping (2.1) defines a 2-norm on* $L_\Phi(X)$

Proof. We need to check that $\|\cdot,\cdot\|_{L_\Phi}$ satisfies the four properties of a 2-norm.

(1) Suppose that $\|f,g\|_{L_\Phi} = 0$. By Lemma 2.3, we obtain

$$\int\limits_X\int\limits_X \Phi\left(\alpha\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right) dx_1dx_2 = 0$$

for every $\alpha > 0$. Since $\Phi\left(\alpha\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right) \geq 0$, we conclude that $\Phi\left(\alpha\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right) = 0$. As a consequence, we have $\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix} = 0$. Hence, f and g are linear dependent. Conversely, suppose $f = kg$ for some $k \in \mathbb{R}$. Observe that

$$\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix} = 0.$$

Then $\|f,g\|_{L_\Phi} =$

$$\inf\left\{b>0 : \frac{1}{2}\int\limits_X\int\limits_X \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|\right) dx_1dx_2 \le 1\right\}$$

$$= \inf\left\{b>0 : \frac{1}{2}\int\limits_X\int\limits_X \Phi(0)\, dx_1dx_2 \le 1\right\} = \inf\{b>0\} = 0.$$

(2) By properties of the determinant, we have $\|f,g\|_{L_\Phi} = \|g,f\|_{L_\Phi}$.

(3) Again, by properties of the determinant, we have $\|\alpha f,g\|_{L_\Phi} = |\alpha|\,\|f,g\|_{L_\Phi}$.

(4) Suppose that $\|f,g+h\|_{L_\Phi} =$

$$\inf\left\{b>0 : \frac{1}{2}\int\limits_X\int\limits_X \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1)+h(x_1) & g(x_2)+h(x_2)\end{pmatrix}\right|\right) dx_1dx_2 \le 1\right\}.$$

Choose $b = \|f,g\|_{L_\Phi} + \|f,h\|_{L_\Phi}$. Using the properties of determinants and Lemma 2.1, we obtain

$$\frac{1}{2}\int\limits_X\int\limits_X \Phi\left(\frac{\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1)+h(x_1) & g(x_2)+h(x_2)\end{pmatrix}\right|}{\|f,g\|_{L_\Phi} + \|f,h\|_{L_\Phi}}\right) dx_1dx_2$$

$$\le \frac{\|f,g\|_{L_\Phi}}{\|f,g\|_{L_\Phi} + \|f,h\|_{L_\Phi}}\int\limits_X\int\limits_X \Phi\left(\frac{\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|}{2\,\|f,g\|_{L_\Phi}}\right) dx_1dx_2$$

$$+\frac{\|f,h\|_{L_\Phi}}{\|f,g\|_{L_\Phi} + \|f,h\|_{L_\Phi}}\int\limits_X\int\limits_X \Phi\left(\frac{\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ h(x_1) & h(x_2)\end{pmatrix}\right|}{2\,\|f,h\|_{L_\Phi}}\right) dx_1dx_2$$

$$\le \frac{\|f,g\|_{L_\Phi}}{\|f,g\|_{L_\Phi} + \|f,h\|_{L_\Phi}} + \frac{\|f,\|_{L_\Phi}}{\|f,g\|_{L_\Phi} + \|f,h\|_{L_\Phi}} = 1.$$

Hence, $\|f,g+h\|_{L_\Phi} \le b = \|f,g\|_{L_\Phi} + \|f,h\|_{L_\Phi}$.

□

The following theorem shows the connection between $L_\Phi(X)$ equipped with $\|\cdot,\cdot\|_{L_\Phi}$ and $L_p(X)$ equipped with $\|\cdot,\cdot\|_{L_p}$.

Theorem 2.5. *If* $\Phi(t) = t^p$ *for* $1 \leq p < \infty$*, then* $\|f,g\|_{L_\Phi} = \|f,g\|_{L_p}$.

Proof. Suppose that $\Phi(t) = t^p$ for $1 \leq p < \infty$. Observe that

$$\begin{aligned}\|f,g\|_{L_\Phi} &= \inf\left\{b>0 : \frac{1}{2}\int\limits_X\int\limits_X \frac{1}{b^p}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|^p dx_1dx_2 \leq 1\right\} \\ &= \inf\left\{b>0 : \frac{1}{2}\int\limits_X\int\limits_X \left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|^p dx_1dx_2 \leq b^p\right\} = \inf A.\end{aligned}$$

Since $\|f,g\|_{L_p}^p = \frac{1}{2}\int\limits_X\int\limits_X \left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|^p dx_1dx_2$, we have $\|f,g\|_{L_p} \leq b$ for every $b \in A$. Consequently, $\|f,g\|_{L_p}$ is lower bound A. Hence, $\|f,g\|_{L_p} \leq \|f,g\|_{L_\Phi}$. Conversely, choosing $b = \|f,g\|_{L_p}$, we have

$$\begin{aligned}&\frac{1}{2}\int\limits_X\int\limits_X \frac{1}{\|f,g\|_{L_p}^p}\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|^p dx_1dx_2 \\ = \;& \frac{1}{\|f,g\|_{L_p}^p}\left(\frac{1}{2}\int\limits_X\int\limits_X \left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ g(x_1) & g(x_2)\end{pmatrix}\right|^p dx_1dx_2\right) = 1.\end{aligned}$$

Hence, $b = \|f,g\|_{L_p} \in A$. Since $\inf A = \|f,g\|_{L_\Phi}$, we have $\|f,g\|_{L_p} \geq \|f,g\|_{L_\Phi}$. Therefore, $\|f,g\|_{L_p} = \|f,g\|_{L_\Phi}$. □

This fact shows that Orlicz spaces equipped a 2-norm can be viewed as generalizations of Lebesgue spaces in 2-normed spaces.

2.2 $L_\Phi(X)$ as a 2-Banach space

We know that $L_\Phi(X)$ is a Banach space with respect to its usual norm $\|\cdot\|_{L_\Phi}$ [12]. Our aim now is to show that $L_\Phi(X)$ is a 2-Banach space with respect to its 2-norm $\|\cdot,\cdot\|_{L_p}$. To do so, we need the following lemmas:

Lemma 2.6. *If* $f, g \in L_\Phi(X)$*, then*

$$\Phi(|f(x)|\,|g(x)|) \leq C\Phi(|f(x)|)\Phi(|g(x)|),$$

for some $C > 0$.

Proof. The proof is by contradiction. If there are no $C > 0$ such that

$$\Phi(|f(x)|\,|g(x)|) \le C\Phi(|f(x)|)\Phi(|g(x)|),$$

then $\Phi(|f(x)|\,|g(x)|) > n\Phi(|f(x)|)\Phi(|g(x)|)$ for all $n \in \mathbb{N}$. That is , $\frac{\Phi(|f(x)||g(x)|)}{n} > \Phi(|f(x)|)\Phi(|g(x)|)$ for all $n \in \mathbb{N}$.. Letting $n \to \infty$, $\frac{\Phi(|f(x)||g(x)|)}{n} \to 0$. So, $\Phi(|f(x)|)\Phi(|g(x)|) < 0$. This contradicts $\Phi(|f(x)|)\Phi(|g(x)|) \ge 0$ for any $x \in X$. Hence $\Phi(|f(x)|\,|g(x)|) \le C\Phi(|f(x)|)\Phi(|g(x)|)$ for $x \in X$ and some $C > 0$. □

By using the above lemma, we have

$$\Phi\left(\frac{|f(x_1)|\,|g(x_2)|}{\|f\|_{L_\Phi}\,\|g\|_{L_\Phi}}\right) \le C_1\Phi\left(\frac{|f(x_1)|}{\|f\|_{L_\Phi}}\right)\Phi\left(\frac{|g(x_2)|}{\|g\|_{L_\Phi}}\right),$$

$$\Phi\left(\frac{|f(x_2)|\,|g(x_1)|}{\|f\|_{L_\Phi}\,\|g\|_{L_\Phi(X)}}\right) \le C_2\Phi\left(\frac{|f(x_2)|}{\|f\|_{L_\Phi}}\right)\Phi\left(\frac{|g(x_1)|}{\|g\|_{L_\Phi}}\right)$$

for some $C_1 > 0$ and $C_2 > 0$. As a consequence, we have the following lemma:

Lemma 2.7. $\|f, g\|_{L_\Phi} \le \frac{\max(C_1,C_2)}{2}\,\|f\|_{L_\Phi}\,\|g\|_{L_\Phi}$.

Moreover, we have the following theorem:

Theorem 2.8. *If a sequence $\{f_n\} \in L_\Phi(X)$ converges to an f with respect to the norm $\|\cdot\|_{L_\Phi}$, then $\{f_n\}$ also converges with respect to the norm $\|\cdot,\cdot\|_{L_\Phi}$. Similarly, if $\{f_n\} \in L_\Phi(X)$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{L_\Phi}$, then $\{f_n\} \in L_\Phi(X)$ is a Cauchy sequence with respect to the norm $\|\cdot,\cdot\|_{L_\Phi}$.*

Proof. Let $\{f_n\} \in L_\Phi(X)$ converge to an f with respect to the norm $\|\cdot\|_{L_\Phi}$ (*i.e.*, $\lim_{n\to\infty}\|f_n - f\|_{L_\Phi} = 0$). Using Lemma 2.7, we obtain

$$\lim_{n\to\infty}\|f_n - f, g\|_{L_\Phi} \le \lim_{n\to\infty}\max(C_1, C_2)\,\|f_n - f\|_{L_\Phi}\,\|g\|_{L_\Phi} = 0$$

for every $g \in L_\Phi(X)$. This shows that $\{f_n\}$ also converges to f with respect to the norm $\|\cdot,\cdot\|_{L_\Phi}$. The proof of the second part is similar. □

Now, we can define a norm that is obtained from the the 2-norm in a certain way. Indeed, if $\{a_1, a_2\}$ is a linearly independent set in $L_\Phi(X)$, then

$$\|f\|^*_{L_\Phi} = \|f, a_1\|_{L_\Phi} + \|f, a_2\|_{L_\Phi} \tag{2.1}$$

defines a norm on $L_\Phi(X)$. Observe that $\|f\|^*_{L_\Phi}$ in (2.1) satisfies the properties of a norm. In particular, we check that if $\|f\|^*_{L_\Phi} = 0$, then $f = 0$ almost everywhere. Indeed, if $\|f\|^*_{L_\Phi} = 0$, then $\|f, a_1\|_{L_\Phi} = 0$ and $\|f, a_2\|_{L_\Phi} = 0$. As a consequence, $f = ka_1$ for some $k \in \mathbb{R}$. Substituting $f = ka_1$, we obtain $k\|a_1, a_2\|_{L_\Phi} = 0$. Since $\|a_1, a_2\|_{L_\Phi} \neq 0$, $k = 0$. Hence, $f = 0$ almost everywhere.

Now, we find the relation between the Banach space with respect the derived norm $\|\cdot\|^*_{L_\Phi(X)}$ and the 2-Banach space with respect to the 2-norm $\|\cdot,\cdot\|_{L_\Phi(X)}$ as follows:

Theorem 2.9. *Let $\{a_1, a_2\}$ be a basis on $L_\Phi(X)$. The Orlicz space $L_\Phi(X)$ with respect to the 2-norm $\|\cdot,\cdot\|_{L_\Phi}$ is a 2-Banach space if and only if $L_\Phi(X)$ with respect to the derived norm $\|\cdot\|^*_{L_\Phi}$ is a Banach space.*

Proof. Assume that $L_\Phi(X)$ with respect to the 2-norm $\|\cdot,\cdot\|_{L_\Phi(X)}$ is a 2-Banach space. If $\{f_n\}$ is the arbitrary Cauchy sequence respect to the norm $\|\cdot\|^*_{L_\Phi}$, then $\|f_m - f_n, a_1\|_{L_\Phi} + \|f_m - f_n, a_2\|_{L_\Phi} = \|f_m - f_n\|^*_{L_\Phi} \to 0$ as $n, m \to \infty$. As a consequence, we obtain $\|f_m - f_n, a_1\|_{L_\Phi} \to 0$ and $\|f_m - f_n, a_2\|_{L_\Phi} \to 0$ as $n, m \to \infty$. Since $\{a_1, a_2\}$ is basis on $L_\Phi(X)$, for every $a \in L_\Phi(X)$ we have

$$\begin{aligned} \|f_m - f_n, a\|_{L_\Phi} &= \|f_m - f_n, \alpha_1 a_1 + \alpha_2 a_2\|_{L_\Phi} \\ &= |\alpha_1| \|f_m - f_n, a_1\|_{L_\Phi} + |\alpha_2| \|f_m - f_n, a_2\|_{L_\Phi}. \end{aligned}$$

This shows that $\|f_m - f_n, a\|_{L_\Phi} \to 0$ as $n, m \to \infty$ for every $a \in L_\Phi(X)$. Hence, $\{f_n\}$ is a Cauchy sequence with respect to the 2-norm. Since $L_\Phi(X)$ is a 2-Banach space, there exists an $f \in L_\Phi(X)$ such that $\|f_n - f, a\|_{L_\Phi} \to 0$ as $n \to \infty$. In particular, we obtain $\|f_n - f, a_1\|_{L_\Phi} \to 0$ and $\|f_n - f, a_2\|_{L_\Phi} \to 0$ as $n \to \infty$. Hence, $\|f_n - f\|^*_{L_\Phi} = \|f_n - f, a_1\|_{L_\Phi} + \|f_n - f, a_2\|_{L_\Phi} \to 0$ as $n \to \infty$. Since the Cauchy sequence $\{f_n\}$ converges to an $f \in L_\Phi(X)$, $L_\Phi(X)$ is a Banach space with respect to the norm $\|\cdot\|^*_{L_\Phi}$.

Conversely, assume that $L_\Phi(X)$ with respect to the norm $\|\cdot\|^*_{L_\Phi}$ is a Banach space. Let $\{f_n\}$ be a Cauchy sequences in $L_\Phi(X)$ with respect to the 2-norm $\|\cdot,\cdot\|_{L_\Phi}$; that is, $\lim_{m,n\to\infty} \|f_m - f_n, a\|_{L_\Phi} = 0$ for every $a \in L_\Phi(X)$. In particular, for $a = a_1$ and $a = a_2$, we obtain $\lim_{m,n\to\infty} \|f_m - f_n, a_1\|_{L_\Phi} = 0$ and $\lim_{m,n\to\infty} \|f_m - f_n, a_2\|_{L_\Phi} = 0$. It follows that

$$\lim_{m,n\to\infty} \|f_m - f_n\|^*_{L_\Phi} = \lim_{m,n\to\infty} \left[\|f_m - f_n, a_1\|_{L_\Phi} + \|f_m - f_n, a_2\|_{L_\Phi}\right] = 0.$$

Hence, $\{f_n\}$ is a Cauchy sequence in $L_\Phi(X)$ with respect to the derived norm $\|\cdot\|^*_{L_\Phi(X)}$. Since $L_\Phi(X)$ is a Banach space with respect to the derived norm

$\|\cdot\|^*_{L_\Phi(X)}$, there is an $f \in L_\Phi(X)$ such that $\lim_{n\to\infty} \|f_n - f\|^*_{L_\Phi(X)} = 0$. As a consequence, we have $\lim_{n\to\infty} \|f_n - f, a_i\|_{L_\Phi(X)} = 0$ for $i = 1, 2$. Since $\{a_1, a_2\}$ is basis on $L_\Phi(X)$, for every $a \in L_\Phi(X)$ we obtain

$$\begin{aligned} \|f_n - f, a\|_{L_\Phi(X)} &= \|f_n - f, \alpha_1 a_1 + \alpha_2 a_2\|_{L_\Phi(X)} \\ &= |\alpha_1| \|f_n - f, a_1\|_{L_\Phi(X)} + |\alpha_2| \|f_n - f, a_2\|_{L_\Phi(X)} . \end{aligned}$$

Hence, $\lim_{n\to\infty} \|f_n - f, a\|_{L_\Phi(X)} = 0$ for every $a \in L_\Phi(X)$. Since the Cauchy sequence $\{f_n\}$ converges to an $f \in L_\Phi(X)$, $L_\Phi(X)$ is a Banach space with respect to the 2-norm $\|\cdot, \cdot\|_{L_\Phi(X)}$. □

3 Concluding Remarks

The following proposition shows the relation between the derived norm $\|\cdot\|^*_{L_\Phi(X)}$ and the usual norm $\|\cdot\|_{L_\Phi(X)}$ on $L_\Phi(X)$.

Proposition 3.1. *Let $\{a_1, a_2\}$ be a linearly independent set in L_Φ. For every $f \in L_\Phi$, we have $\|f\|^*_{L_\Phi} \leq \max(C_1, C_2)(\|a_1\|_{L_\Phi} + \|a_2\|_{L_\Phi}) \|f\|_{L_\Phi}$.*

Proof. Using Lemma 2.7, we have $\|f, a_1\|_{L_\Phi} \leq \frac{\max(C_1,C_2)}{2} \|f\|_{L_\Phi} \|a_1\|_{L_\Phi}$ and $\|f, a_2\|_{L_\Phi} \leq \frac{\max(C_1,C_2)}{2} \|f\|_{L_\Phi} \|a_2\|_{L_\Phi}$ for every $f \in L_\Phi(X)$. By (2.1), we have $\|f\|^*_{L_\Phi} \leq \max(C_1, C_2)(\|a_1\|_{L_\Phi} + \|a_2\|_{L_\Phi}) \|f\|_{L_\Phi}$. □

As a result of Proposition 3.1 and Lemma 2.7, we have

Corollary 3.2. *If a sequence $\{f_n\} \in L_\Phi(X)$ converges to an f with respect to $\|\cdot\|_{L_\Phi}$, then $\{f_n\}$ also converges with respect to $\|\cdot\|^*_{L_\Phi}$. Similarly, if $\{f_n\} \in L_\Phi(X)$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{L_\Phi}$, then $\{f_n\} \in L_\Phi(X)$ is a Cauchy sequence with respect to the norm $\|\cdot\|^*_{L_\Phi}$.*

Unfortunately, up to now, we have not been able to prove that the derived norm $\|\cdot\|^*_{L_\Phi}$ and the usual norm $\|\cdot\|_{L_\Phi}$ on $L_\Phi(X)$ are equivalent. Therefore, we do not know if $L_\Phi(X)$ is a Banach space with respect to the derived norm $\|\cdot\|^*_{L_\Phi}$ using a linearly independent set $\{a_1, a_2\}$ in $L_\Phi(X)$.

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