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Prime fuzzy Bd-ideals of Bd-algebras

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Abstract

In [7], Nakkhasen et al. characterized the notion of fuzzy Bd-ideals in Bd-algebras. In this article, we introduce the concepts of prime (resp. semiprime) subsets and prime (resp. semiprime) fuzzy sets in Bd-algebras. Then, we study the relationships between prime (resp. semiprime) Bd-ideals and prime (resp. semiprime) fuzzy Bd-ideals of Bd-algebras.

1 Introduction

In 1965, Zadeh [11] introduced the concept of fuzzy sets. When handling uncertainty in real-world circumstances, this idea is quite helpful. In 1999, Neggers and Kim [8] introduced the notion of d-algebras as an algebraic structure that generalizes BCK-algebras. and, in 2002, they introduced [9]

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the notion of B-algebras which is a class of abstract algebraic structures. Akram and Dar [2] presented the concepts of fuzzy d-ideals and fuzzy subalgebras of d-algebras and they also explored some of their features. Then, Ahn and So [1] studied the notions of normal fuzzy d-ideals, maximal fuzzy d-ideals and completely normal fuzzy d-ideals in d-transitive d-algebras. Afterwards, Khalil [6] introduced a new class of fuzzy d-algebras called Γ-fuzzy d-algebras. Meanwhile, the concept of fuzzy sets was also investigated in B-algebras as well. Jun et al. [5] presented the concepts of fuzzy B-subalgebras and fuzzy normals of B-algebras and examined fuzzy B-subalgebras in B-algebras. Subsequently, as a generalization of fuzzy B-subalgebras in B-algebras, Baghini and Saeid [3] defined the idea of (α, β)-fuzzy B-algebras. In hyper B-algebras, Tabaranza and Vilela [10] investigated the relationships of fuzzy hyper B-ideals, fuzzy weak hyper B-ideals and fuzzy strong hyper B-ideals.

In 2022, Bantaojai and coworkers [4] defined a new algebraic structure called Bd-algebras by combining some properties of B-algebras and d-algebras. Recently, in 2024, Nakkhasen et al. [7] have brought the concept of fuzzy sets to study in Bd-algebras, where they have introduced the concept of fuzzy Bd-ideals and discussed its various properties. In this paper, we introduce the concepts of prime (resp. semiprime) subsets and prime (resp. semiprime) fuzzy sets in Bd-algebras. Moreover, we study the relationships between prime (resp. semiprime) Bd-ideals and prime (resp. semiprime) fuzzy Bd-ideals of Bd-algebras.

2 Preliminaries

A *fuzzy set* [11] μ is a mapping $X \to [0,1]$, where X is a nonempty set and $[0,1]$ is an unit interval. For any fuzzy sets μ and λ of a nonempty set X, we use the following notations:
(i) $\mu^c(x) = 1 - \mu(x)$, for all $x \in X$; (ii) $(\mu \cap \lambda)(x) = \min\{\mu(x), \lambda(x)\}$, for all $x \in X$; (iii) $(\mu \cup \lambda)(x) = \max\{\mu(x), \lambda(x)\}$, for all $x \in X$.

Let A be any subset of a nonempty set X. Then, a fuzzy set χ_A of X is defined, for every $x \in X$, as

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Let μ be any fuzzy set of a nonempty set X and let $t \in [0,1]$. Then
(i) the set $U(\mu, t) = \{x \in X \mid \mu(x) \geq t\}$ is called an *upper t-level subset of*

μ; (ii) the set $U^+(\mu, t) = \{x \in X \mid \mu(x) > t\}$ is called an *upper t-strong level subset of* μ; (iii) the set $L(\mu, t) = \{x \in X \mid \mu(x) \leq t\}$ is called a *lower t-level subset of* μ; (iv) the set $L^-(\mu, t) = \{x \in X \mid \mu(x) < t\}$ is called a *lower t-strong level subset of* μ.

Definition 2.1. *[4] An algebraic structure $(X, *, 0)$ is said to be a Bd-algebra if it satisfies the following conditions, for any $x, y \in X$:*
(i) $x * 0 = x$;
(ii) if $x * y = 0$ and $y * x = 0$, then $x = y$.

Definition 2.2. *[4] Let I be a nonempty subset of a Bd-algebra $(X, *, 0)$. Then I is called a Bd-ideal of X if it satisfies the following properties:*
(i) $0 \in I$;
(ii) if for all $x, y \in X$, $x * y \in I$ and $y \in I$, then $x \in I$; (iii) for every $x \in I$ and $y \in X$, $x * y \in I$.

Definition 2.3. *[7] Let $(X, *, 0)$ be a Bd-algebra. A fuzzy set μ of X is called a fuzzy Bd-ideal of X if, for each $x, y \in X$, it satisfies the following conditions:*
(i) $\mu(0) \geq \mu(x)$;
(ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$; (iii) $\mu(x * y) \geq \mu(x)$.

For the sake of simplicity, we denote a Bd-algebra $(X, *, 0)$ by the nonempty set X. The following results were presented by Nakkhasen et al. [7] as follows:

Lemma 2.4. *[7] Let A be a nonempty subset of a Bd-algebra X. Then A is a Bd-ideal of X if and only if χ_A is a fuzzy Bd-ideal of X.*

Lemma 2.5. *[7] Let μ be a fuzzy set of a Bd-algebra X, Then μ is a fuzzy Bd-ideal of X if and only if for every $t \in [0, 1]$, $U(\mu, t) \neq \emptyset$ is a Bd-ideal of X.*

Lemma 2.6. *[7] Let μ be a fuzzy set of a Bd-algebra X, Then μ is a fuzzy Bd-ideal of X if and only if for every $t \in [0, 1]$, $U^+(\mu, t) \neq \emptyset$ is a Bd-ideal of X.*

Lemma 2.7. *[7] Let μ be a fuzzy set of a Bd-algebra X. Then μ^c is a fuzzy Bd-ideal of X if and only if, for every $t \in [0, 1]$, $L(\mu, t) \neq \emptyset$ is a Bd-ideal of X.*

Lemma 2.8. *[7] Let μ be a fuzzy set of a Bd-algebra X. Then μ^c is a fuzzy Bd-ideal of X if and only if, for every $t \in [0, 1]$, $L^-(\mu, t) \neq \emptyset$ is a Bd-ideal of X.*

3 Prime fuzzy *Bd*-ideals of *Bd*-algebras

In this section, we introduce the concepts of prime (resp. semiprime) subsets and prime (resp. semiprime) fuzzy sets in Bd-algebras. Then we investigate the connections between prime (resp. semiprime) Bd-ideals and prime (resp. semiprime) fuzzy Bd-ideals of Bd-algebras.

Definition 3.1. *Let X be a Bd-algebra and let A be a nonempty subset of X. Then A is called:*

(i) prime if for any $x, y \in X$, if $x * y \in A$, then $x \in A$ or $y \in A$;

(ii) semiprime if for any $x \in X$, if $x * x \in A$, then $x \in A$.

Definition 3.2. *Let X be a Bd-algebra and let μ be a fuzzy set of X. Then μ is said to be:*

(i) prime if $\mu(x * y) \leq \max\{\mu(x), \mu(y)\}$, for all $x, y \in X$;

(ii) semiprime if $\mu(x * x) \leq \mu(x)$, for all $x \in X$.

It is known that every prime fuzzy set of a Bd-algebra X is also a semiprime fuzzy set of X. In general, the converse of this statement is not true as the following example shows:

Example 3.3. *Consider the set $X = \{0, a, b, c\}$ with the binary operation $*$ on X as follows:*

$*$	0	a	b	c
0	0	a	a	a
a	a	a	a	a
b	b	b	b	b
c	c	a	a	c

We can see that $(X, *, 0)$ is a Bd-algebra. Now, the fuzzy sets μ and λ of X are defined by $\mu(0) = 0.8, \mu(a) = 0.3, \mu(b) = 0.6, \mu(c) = 0.4$, and $\lambda(0) = 0.9, \lambda(a) = 0.7, \lambda(b) = 0.4, \lambda(c) = 0.4$. We can carefully calculate that μ is a prime fuzzy set of X. Moreover, λ is a semiprime fuzzy set of X, but it is not a prime fuzzy set of X because $\lambda(c * b) > \max\{\lambda(c), \lambda(b)\}$.

Theorem 3.4. *Let X be a Bd-algebra and let A be a nonempty subset of X. Then A is a prime Bd-ideal of X if and only if χ_A is a prime fuzzy Bd-ideal of X.*

Proof. Assume that A is a prime Bd-ideal of X. By Lemma 2.4, we have χ_A is a fuzzy Bd-ideal of X. Let $x, y \in X$. If $x * y \in A$, then $x \in A$ or $y \in A$. It follows that $\chi_A(x * y) = 1 = \max\{\chi_A(x), \chi_A(y)\}$. If $x * y \notin A$, then we have $\chi_A(x * y) = 0 \leq \max\{\chi_A(x), \chi_A(y)\}$. Hence, χ_A is a prime fuzzy Bd-ideal of X. Conversely, assume that χ_A is a prime fuzzy Bd-ideal of X. By Lemma 2.4, we have A is a Bd-ideal of X. Let $x, y \in X$ be such that $x * y \in A$. Then $1 = \chi_A(x * y) \leq \max\{\chi_A(x), \chi_A(y)\}$. Thus $\max\{\chi_A(x), \chi_A(y)\} = 1$. This implies that $\chi_A(x) = 1$ or $\chi_A(y) = 1$; that is, $x \in A$ or $y \in A$. Therefore, A is a prime Bd-ideal of X. □

Theorem 3.5. *Let X be a Bd-algebra and let μ be a fuzzy set of X. Then μ is a prime fuzzy Bd-ideal of X if and only if for any $t \in [0, 1]$, $U(\mu, t)$ is a prime Bd-ideal of X when it is nonempty.*

Proof. Assume that μ is a prime fuzzy Bd-ideal of X. Then μ is a fuzzy Bd-ideal of X. Let $t \in [0, 1]$ be such that $U(\mu, t) \neq \emptyset$. By Lemma 2.5, we have $U(\mu, t)$ is a Bd-ideal of X. Let $x, y \in X$ be such that $x*y \in U(\mu, t)$. Suppose that $x \notin U(\mu, t)$ and $y \notin U(\mu, t)$. Then $\mu(x) < t$ and $\mu(y) < t$. Since μ is prime, we have $\mu(x * y) \leq \max\{\mu(x), \mu(y)\} < t$. Also, $x * y \notin U(\mu, t)$, which is a contradiction. Hence, $x \in U(\mu, t)$ or $y \in U(\mu, t)$. Therefore, $U(\mu, t)$ is a prime fuzzy Bd-ideal of X.

Conversely, assume that, for any $t \in [0, 1]$, $U(\mu, t)$ is a prime Bd-ideal of X if it is nonempty. This implies that the nonempty set $U(\mu, t)$ is a Bd-ideal of X for all $t \in [0, 1]$. By Lemma 2.5, we have μ is a fuzzy Bd-ideal of X. Next, let $x, y \in X$. Put $t = \mu(x * y)$. Thus $x * y \in U(\mu, t) \neq \emptyset$. By assumption, we have $U(\mu, t)$ is prime. It turns out that $x \in U(\mu, t)$ or $y \in U(\mu, t)$. So $\mu(x) \geq t$ or $\mu(y) \geq t$. Hence $\mu(x * y) = t \leq \max\{\mu(x), \mu(y)\}$. We conclude that μ is a prime fuzzy Bd-ideal of X. □

Theorem 3.6. *Let X be a Bd-algebra and let μ be a fuzzy set of X. Then μ is a prime fuzzy Bd-ideal of X if and only if, for any $t \in [0, 1]$, $U^+(\mu, t)$ is a prime Bd-ideal of X when it is nonempty.*

Proof. Assume that μ is a prime fuzzy Bd-ideal of X. Then μ is a fuzzy Bd-ideal of X. Let $t \in [0, 1]$ be such that $U^+(\mu, t) \neq \emptyset$. Then $U^+(\mu, t)$ is a Bd-ideal of X by Lemma 2.6. Now, let $x, y \in X$ be such that $x*y \in U^+(\mu, t)$. Suppose that $x \notin U^+(\mu, t)$ and $y \notin U^+(\mu, t)$. Then $\mu(x) \leq t$ and $\mu(y) \leq t$. Since μ is prime, we have $\mu(x * y) \leq \max\{\mu(x), \mu(y)\} \leq t$. This implies that $x*y \notin U^+(\mu, t)$ which is a contradiction. Hence $x \in U^+(\mu, t)$ or $y \in U^+(\mu, t)$. Therefore, $U^+(\mu, t)$ is a prime Bd-ideal of X.

Conversely, assume that for any $t \in [0,1]$, $U^+(\mu,t)$ is a prime Bd-ideal of X if it is nonempty. We obtain that the nonempty set $U^+(\mu,t)$ is also a Bd-ideal of X, for all $t \in [0,1]$. By Lemma 2.6, we get μ is a fuzzy Bd-ideal of X. Now, let $x,y \in X$. Suppose that $\mu(x*y) > \max\{\mu(x),\mu(y)\}$. Take $t = \max\{\mu(x),\mu(y)\}$. It follows that $x*y \in U^+(\mu,t)$. By the given assumption, we have $U^+(\mu,t)$ is prime. This implies that $x \in U^+(\mu,t)$ or $y \in U^+(\mu,t)$; that is, $\mu(x) > t = \max\{\mu(x),\mu(y)\}$ or $\mu(y) > t = \max\{\mu(x),\mu(y)\}$. This is a contradiction. Hence $\mu(x*y) \le \max\{\mu(x),\mu(y)\}$. This shows that μ is a prime fuzzy Bd-ideal of X. □

Theorem 3.7. *Let X be a Bd-algebra and let μ be a fuzzy set of X. Then μ^c is a prime fuzzy Bd-ideal of X if and only if for every $t \in [0,1]$, $L(\mu,t)$ is a prime Bd-ideal of X when it is nonempty.*

Proof. Assume that μ^c is a prime fuzzy Bd-ideal of X. Then μ^c is a fuzzy Bd-ideal of X. Let $t \in [0,1]$ be such that $L(\mu,t) \neq \emptyset$. By Lemma 2.7, we have $L(\mu,t)$ is a Bd-ideal of X. Let $x,y \in X$ be such that $x*y \in L(\mu,t)$. Suppose that $x \notin L(\mu,t)$ and $y \notin L(\mu,t)$. So $\mu(x) > t$ and $\mu(y) > t$. Since μ^c is prime, we get $1-\mu(x*y) = \mu^c(x*y) \le \max\{\mu^c(x),\mu^c(y)\} = \max\{1-\mu(x),1-\mu(y)\} = 1-\min\{\mu(x),\mu(y)\}$. This implies that $\mu(x*y) \ge \min\{\mu(x),\mu(y)\} > t$. That is, $x*y \notin L(\mu,t)$ which is a contradiction. Hence $x \in L(\mu,t)$ or $y \in L(\mu,t)$. Therefore, $L(\mu,t)$ is a prime Bd-ideal of X.

Conversely, assume that for every $t \in [0,1]$, $L(\mu,t)$ is a prime Bd-ideal of X if it is nonempty. By Lemma 2.7, we have μ^c is a fuzzy Bd-ideal of X. Now, let $x,y \in X$. Put $\mu(x*y) = t$. Then $x*y \in L(\mu,t)$. By assumption, we have $L(\mu,t)$ is prime. Thus $x \in L(\mu,t)$ or $y \in L(\mu,t)$. We obtain $\mu(x) \le t$ or $\mu(y) \le t$. It follows that $\mu(x*y) = t \ge \max\{\mu(x),\mu(y)\}$. This means that $\mu^c(x*y) = 1-\mu(x*y) \le 1-\min\{\mu(x),\mu(y)\} = \max\{1-\mu(x),1-\mu(y)\} = \max\{\mu^c(x),\mu^c(y)\}$. Consequently, μ^c is a prime fuzzy Bd-ideal of X. □

Theorem 3.8. *Let X be a Bd-algebra and let μ be a fuzzy set of X. Then μ^c is a prime fuzzy Bd-ideal of X if and only if for every $t \in [0,1]$, $L^-(\mu,t)$ is a prime Bd-ideal of X when it is nonempty.*

Proof. Assume that μ^c is a prime fuzzy Bd-ideal of X. Then μ^c is a fuzzy Bd-ideal of X. Let $t \in [0,1]$ be such that $L^-(\mu,t) \neq \emptyset$. We get $L^-(\mu,t)$ is a Bd-ideal of X by Lemma 2.8. Next, let $x,y \in X$ be such that $x*y \in L^-(\mu,t)$. Suppose that $x \notin L^-(\mu,t)$ and $y \notin L^-(\mu,t)$. So $\mu(x) \ge t$ and $\mu(y) \ge t$. Since μ^c is prime, we have $1-\mu(x*y) = \mu^c(x*y) \le \max\{\mu^c(x),\mu^c(y)\} = \max\{1-\mu(x),1-\mu(y)\} = 1-\min\{\mu(x),\mu(y)\}$. This implies that $\mu(x*y) \ge$

$\min\{\mu(x), \mu(y)\} \geq t$, and so $x * y \notin L^-(\mu, t)$. This is a contradiction. Hence $x \in L^-(\mu, t)$ or $y \in L^-(\mu, t)$. Therefore, $L^-(\mu, t)$ is a Bd-ideal of X.

Conversely, assume that for every $t \in [0, 1]$, $L^-(\mu, t)$ is a prime Bd-ideal of X if it is nonempty. This implies that for every $t \in [0, 1]$, $L^-(\mu, t)$ is a Bd-ideal of X if it is nonempty. By Lemma 2.8, we have μ^c is a fuzzy Bd-ideal of X. Now, let $x, y \in X$. Suppose that $\mu^c(x * y) > \max\{\mu^c(x), \mu^c(y)\}$. Then $1 - \mu(x * y) > \max\{1 - \mu(x), 1 - \mu(y)\} = 1 - \min\{\mu(x), \mu(y)\}$. Also, $\mu(x * y) < \min\{\mu(x), \mu(y)\}$. Thus, letting $t = \min\{\mu(x), \mu(y)\}$, we obtain $x * y \in L^-(\mu, t)$; that is, $L^-(\mu, t) \neq \emptyset$. By the hypothesis, we have $L^-(\mu, t)$ is prime. It follows that $x \in L^-(\mu, t)$ or $y \in L^-(\mu, t)$. It turns out that $\mu(x) < t = \min\{\mu(x), \mu(y)\}$ or $\mu(y) < t = \min\{\mu(x), \mu(y)\}$ which is a contradiction. Hence, $\mu^c(x * y) \leq \max\{\mu^c(x), \mu^c(y)\}$. This shows that μ^c is a prime fuzzy Bd-ideal of X. □

The following results can be proven similar to Theorems 3.4, 3.5, 3.6, 3.7 and 3.8, respectively.

Theorem 3.9. *Let X be a Bd-algebra and let A be a nonempty subset of X. Then A is a semiprime Bd-ideal of X if and only if χ_A is a semiprime fuzzy Bd-ideal of X.*

Theorem 3.10. *Let X be a Bd-algebra and let μ be a fuzzy set of X. Then μ is a semiprime fuzzy Bd-ideal of X if and only if for any $t \in [0, 1]$, $U(\mu, t)$ is a semiprime Bd-ideal of X when it is nonempty.*

Theorem 3.11. *Let X be a Bd-algebra and let μ be a fuzzy set of X. Then μ is a semiprime fuzzy Bd-ideal of X if and only if for any $t \in [0, 1]$, $U^+(\mu, t)$ is a semiprime Bd-ideal of X when it is nonempty.*

Theorem 3.12. *Let X be a Bd-algebra and let μ be a fuzzy set of X. Then μ^c is a semiprime fuzzy Bd-ideal of X if and only if for every $t \in [0, 1]$, $L(\mu, t)$ is a semiprime Bd-ideal of X when it is nonempty.*

Theorem 3.13. *Let X be a Bd-algebra and let μ be a fuzzy set of X. Then μ^c is a semiprime fuzzy Bd-ideal of X if and only if for every $t \in [0, 1]$, $L^-(\mu, t)$ is a semiprime Bd-ideal of X when it is nonempty.*

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