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The Operating-Characteristic Curve for $\bar{x}$ Control Chart when the Normality Assumption is Violated

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Abstract

Control charts are known to be powerful tools for monitoring the quality of processes in many industries. For specified control charts, the operating-characteristic (OC) curve shows the probability of failing to detect a shift of a particular size. In this paper, we investigate the OC curves of the well-known Shewhart $\bar{x}$ chart when the normality assumption is violated. The small size of shift is more sensitive to departures from normality than the large one.

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1 Introduction

Control charts are typically used in establishing whether a process is in a state of statistical control or not. In practice, there are two distinct phases of control charting and each case has unique control limit specifications. Lowry and Montgomery [1] stated that Phase I consists of using the charts for retrospectively testing whether the process was in control when the first m preliminary subgroups were being drawn and the sample statistics computed. The objective is to obtain an in-control set of data to establish control limits for future monitoring purposes. These control limits are used in Phase II to test whether the process remains in-control when future subgroups are drawn during the second phase. Therefore, Phase II consists of using the control chart to detect any departures of the underlying process and relies on the assumption that the in-control parameters are known.

The most known and commonly-used control charts are Shewhart control charts which are capable of quickly detecting shifts in the testing process that are larger than 1.5σ [2], but they are much less likely to be effective in Phase II because they are not very sensitive to small and moderate size process shifts [3]. The $\bar{x}$ quality control chart is a type of Shewhart control chart that used to monitor the mean of a process based on samples taken in a given time. The control limits are used to monitor the mean of the process going forward. If a point is out of the control limits, it indicates that the mean of the process is out-of-control; assignable causes may be suspected at this point.

The performance of a control chart is an important consideration and one of the popular measure of a control chart's performance is the operating-characteristic (OC) function which is a measure of how quickly a chart will detect or react to a change in the process and can be used to compare control charting plans. That measure also provides useful information about the operational performance of a control chart [4].

There is often an assumption that links normality and control chart in the development of the performance properties of $\bar{x}$ control chart; that is, that the underlying distribution of the quality characteristic is normal. In many situations, we may have reason to doubt the validity of this assumption. For example, we may know that the underlying distribution is not normal because we have collected extensive data that indicate the normality assumption is inappropriate. Nidsunkid *et al.* [5] studied the effects of violations of the multivariate normality assumption in multivariate Shewhart and MEWMA control charts when the random vector ($\boldsymbol{X}$) is from the multivariate normal, multivariate t, multivariate uniform, multivariate beta and multivariate lognormal distributions. Nidsunkid *et al.* [6] studied the performance of control chart for MCUSUM control charts when the multivariate normality assumption is violated. In addition the impact of a random vector with variables from normal and non-normal distributions on multivariate control charts was proposed by Nidsunkid *et al.* [7]. The type of skewness for a distribution, even sampling data are from finite population [8], all affect the performance of control chart and control chart based on regression adjustment [9] in different ways.

In this article, we examine the OC curve for $\bar{x}$ control chart when the quality characteristic is either a normal distribution or a non-normal one.

2 The $\bar{x}$ Control Chart

Control of the process average or mean quality level is usually done with the control chart for means, or the $\bar{x}$ control chart. It is assumed that the process to be monitored yields some quality characteristic values, X_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ that are normally distributed with in-control mean (μ) and standard deviation (σ), where m samples are available each containing n observations on the quality characteristics. Typically, a lower control limit (LCL) and an upper control limit (UCL) for the quality characteristics are required to construct the control chart. The control limits for the $\bar{x}$ chart when values for μ and σ are given are

$$\begin{aligned} UCL &= \mu + \frac{3}{\sqrt{n}}\sigma \\ Center\ line &= \mu \\ LCL &= \mu - \frac{3}{\sqrt{n}}\sigma \end{aligned} \tag{2.1}$$

3 The Operating-Characteristic Function

The ability of the $\bar{x}$ charts to detect shifts in process quality is described by their operating characteristic (OC) curves. Consider the OC curve for an $\bar{x}$ control chart, the standard deviation σ is assumed known and constant. If the mean shifts from the in-control value (μ_0) to another value $\mu_1 = \mu_0 + k\sigma$, the probability of not detecting this shift on the first subsequent sample or the β-risk is

$$\beta = P\{LCL \leq \bar{x} \leq UCL | \mu = \mu_1 = \mu_0 + k\sigma\}. \tag{3.2}$$

To construct the OC curve for the $\bar{x}$ control chart, plot the β-risk against the magnitude of the shift (k) we wish to detect expressed in standard deviation units for various sample sizes n.

4 Methodology

In this article, we construct the OC curve for the $\bar{x}$ control chart when the distribution of quality characteristics (variables) are normal, t (more heavy–tailed) and lognormal (skewed-right) distributions. The β-risk against the magnitude of the shift (k) are plotted for various sample sizes n. The probabilities may be evaluated directly from equation (3.2) for the case of three-sigma limits.

5 Results

Figures 1, 2 and 3 display the OC curves for the $\bar{x}$ chart with three-sigma limits when the quality characteristics are normal, t and lognormal distributions, respectively. If the quality characteristic is normal with small sample sizes, the $\bar{x}$ chart is not particularly effective in detecting a small shift. For instance, the β-risk or the probability of not detecting at shifts $0.0 < k < 1.0$ are around 0.9972 to 0.9821, 0.9970 to 0.9579 and 0.9969 to 0.9252 when $n = 1$, 2 and 3, respectively. For t quality characteristic, the probabilities of not detecting at tiny shifts are smaller than normal quality characteristic for all sample sizes n. In addition, the β-risk of lognormal variables with small n are less than normal variables for small and moderate shifts.

Table 1: The values of $\beta(1-\beta)$ and $\beta^2(1-\beta)$ where shift is 4.0σ.

	$\beta(1-\beta)$			$\beta^2(1-\beta)$		
	normal	t	lognormal	normal	t	lognormal
n=1	0.1335	0.0821	0.1163	0.0212	0.0074	0.0156
n=2	0.0039	0.0223	0.0474	0.0000	0.0005	0.0024
n=3	0.0000	0.0134	0.0311	0.0000	0.0002	0.0010
n=4	0.0000	0.0100	0.0240	0.0000	0.0001	0.0006
n=5	0.0000	0.0082	0.0199	0.0000	0.0001	0.0004
n=10	0.0000	0.0049	0.0121	0.0000	0.0000	0.0001
n=15	0.0000	0.0038	0.0094	0.0000	0.0000	0.0001
n=20	0.0000	0.0032	0.0079	0.0000	0.0000	0.0001

The probability that the shift will be detected on the first subsequent sample is $1 - \beta$. If the size of shift is large, the probability that the shift is detected on the first sample will be close to 1 even though the variable distribution is not normal. For instance, if the shift is 4.8σ and $n = 2$, then we have $1-\beta = 1-0.001 = 0.9999$, $1-0.0088 = 0.9912$ and $1-0.0211 = 0.9789$, approximately for normal, t, and lognormal, respectively. Moreover, the probability that the shift is detected on the second sample is $\beta(1-\beta)$ whereas the probability that it is detected on the third sample is $\beta^2(1-\beta)$. Therefore, the probability that the process will be detected on the rth subsequent sample is $\beta^{r-1}(1-\beta)$. When the size of shift is large, the $\beta(1-\beta)$ and $\beta^2(1-\beta)$ for non-normal distribution tend to be larger than those for normal distribution. In Table 1, we show the pattern of $\beta(1-\beta)$ and $\beta^2(1-\beta)$ where the shift is 4.0σ.

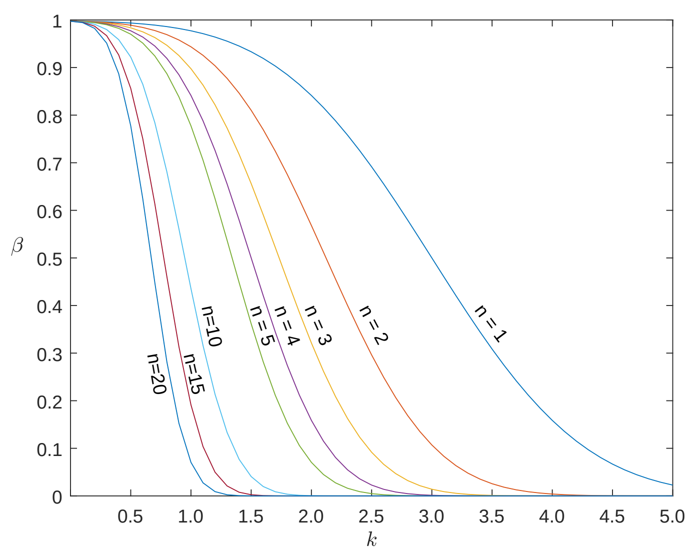

Figure 1: OC curve for the $\bar{x}$ chart with three-sigma limits and quality characteristic is normal distribution

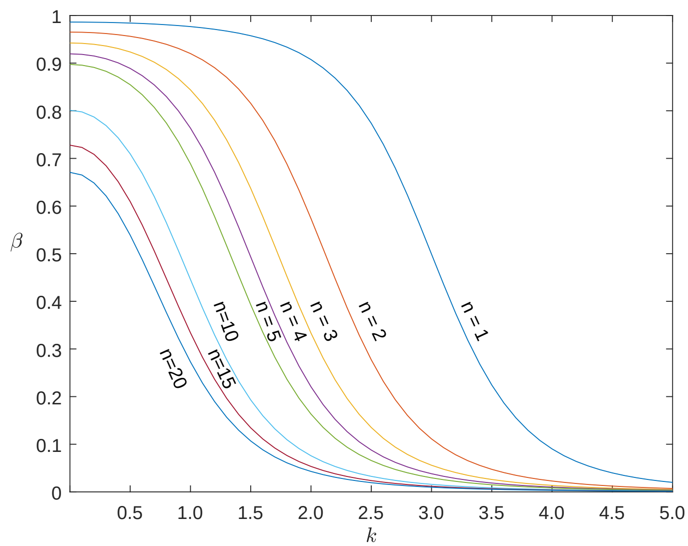

Figure 2: OC curve for the $\bar{x}$ chart with three-sigma limits and quality characteristic is t (more heavy–tailed) distribution

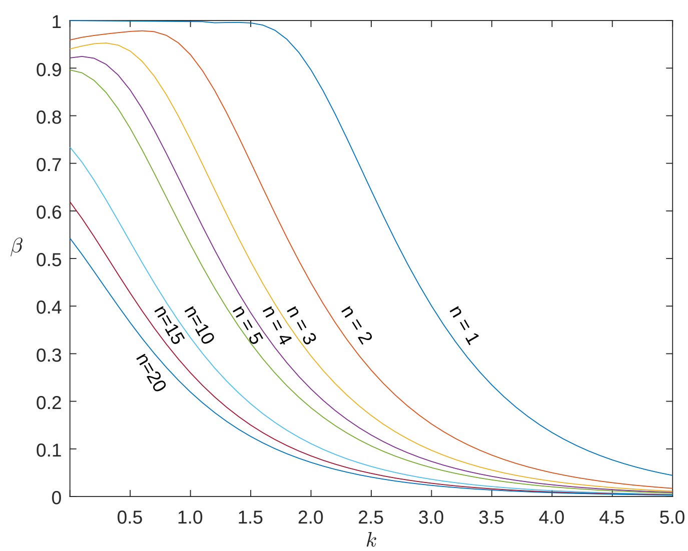

Figure 3: OC curve for the $\bar{x}$ chart with three-sigma limits and quality characteristic is lognormal (skewed-right) distribution.

6 Conclusions

The departures from normality on the control chart for $\bar{x}$ obviously affect the OC curve. In this paper, we indicated that for heavy-tailed and skewed-right variable, the probabilities of not detecting small shifts are less than normal variable for small sizes n. However, if the size of shift is larger, then the probability that the shift is detected on the first sample will be close to 1 for all distributions.

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