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The unit solutions of $x + y = xy$ over quadratic number fields

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Abstract

In this article, we show that the equation $xy = x + y$ has a unit solution over the ring of integers of a quadratic field $\mathbb{Q}(\sqrt{d})$ if and only if $d = -3$ or $d = 5$.

1 Introduction

For a square free integer d, the unit solutions of the equation $xyz = x+y+z$ over the ring of integers of a quadratic field $\mathbb{Q}(\sqrt{d})$ were found in 1987 by Mollin, Small, Varadarajan, and Wash [3]. They showed that when $d = -1, 2,$ or 5, there is a finite number of unit solutions to the equation $xyz = x+y+z$ and that for other values of d there are no solutions. Inspired by a study of the unit solutions of the equation $xyz = x + y + z$ over the ring of integers of a quadratic field, it is natural to ask a similar question to the equation $xy = x + y$. We will show that the equation $xy = x + y$ has a unit solution over the ring of integers of a quadratic field $\mathbb{Q}(\sqrt{d})$ if and only if $d = -3$ or $d = 5$.

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2 Main result

Theorem 2.1. *Let d be a square free integer and let $S(d)$ be a set of unit solutions of the equation $xy = x+y$ over the ring of integers of the quadratic fields $\mathbb{Q}(\sqrt{d})$. Then*

$$S(d) = \begin{cases} \emptyset & \text{if } d \neq -3, 5 \\ \{\left(\frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}\right), \left(\frac{1-\sqrt{-3}}{2}, \frac{1+\sqrt{-3}}{2}\right)\} & \text{if } d = -3 \\ \{\left(\frac{3\pm\sqrt{5}}{2}, \frac{1\pm\sqrt{5}}{2}\right), \left(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right), \left(\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right), & \\ \left(\frac{1\pm\sqrt{5}}{2}, \frac{3\pm\sqrt{5}}{2}\right)\} & \text{if } d = 5. \end{cases}$$

Proof. Assume that x and y are units and $xy = x+y$. Then $(x-1)(y-1) = 1$. It follows that $N(x-1) = N(y-1) = \pm 1$.
Case 1 $d \equiv 2, 3 \mod 4$. Thus $\mathcal{O}_K = \mathbb{Z}[\sqrt{d}]$. Let $x = a + b\sqrt{d}$ for some $a, b \in \mathbb{Z}$. Then $x - 1 = a - 1 + b\sqrt{d}$ for some $a, b \in \mathbb{Z}$. Then $N(x) = a^2 - b^2 d$ and $N(x-1) = (a-1)^2 - b^2 d$. If $N(x) = N(x-1)$, then $a^2 - b^2 d = (a-1)^2 - b^2 d$. Thus $|a| = |a+1|$, which is a contradiction. If $N(x) = -N(x-1)$, then $a^2 - b^2 d = -(a-1)^2 + b^2 d$ and this implies that $2a^2 - 2a - 2b^2 d = -1$, which is also a contradiction.
Case 2 $d \equiv 1 \mod 4$. Thus $\mathcal{O}_K = \mathbb{Z}[(1+\sqrt{d})/2]$. Let $x = a + b(1+\sqrt{d})/2$ for some $a, b \in \mathbb{Z}$. Then $y = \frac{1}{a-1+b\left(\frac{1+\sqrt{d}}{2}\right)} + 1 = \frac{a-1+b\left(\frac{1-\sqrt{d}}{2}\right)}{N(x-1)} + 1$.

Case 2.1 $N(x) = N(x-1) = 1$. We obtain $b = 1 - 2a$. Thus $N(x) = a^2 + a(1-2a) + (1-2a)^2\left(\frac{1-d}{4}\right) = 1$. This implies that $a = \frac{1-4M\pm\sqrt{12M-3}}{2(1-4M)}$ where $M = (1-d)/4$. Thus $12M - 3 = z^2$ for some $z \in \mathbb{Z}$. This yields $3d = -z^2$. So $z = 3z_0$ for some integer z_0. We get $d = -3z_0^2$. Since d is a square free integer, $d = -3$. It follows that $a = 0$ and $b = 1$ or $a = 1$ and $b = -1$. Consequently, $x = \frac{1\pm\sqrt{-3}}{2}$ and $y = \frac{1\mp\sqrt{-3}}{2}$.

Case 2.2 $N(x) = 1$ but $N(x-1) = -1$. We obtain $b = 3 - 2a$ and $N(x) = a^2 + a(3-2a) + (3-2a)^2((1-d)/4) = 1$. This implies that $a = \frac{3(1-4M)\pm\sqrt{5-20M}}{2(1-4M)}$ where $M = (1-d)/4$. Thus $5 - 20M = z^2$ for some $z \in \mathbb{Z}$. This yields $5d = z^2$. So $z = 5z_0$ for some integer z_0. We get $d = 5z_0^2$. Since d is a square free integer, we have $d = 5$. It follows that $a = 1$ and $b = 1$ or $a = 2$ and $b = -1$. Thus, solutions x and y are $x = \frac{3\pm\sqrt{5}}{2}, y = \frac{1\pm\sqrt{5}}{2}$.

Case 2.3 $N(x) = -1$ but $N(x-1) = 1$. Thus $b = -1 - 2a$ and this implies that $N(x) = a^2 + a(-1-2a) + (-1-2a)^2((1-d)/4) = -1$. Hence

$a = \frac{4M-1\pm\sqrt{5-20M}}{2(1-4M)}$ where $M = (1-d)/4$. Thus $5-20M = z^2$ for some $z \in \mathbb{Z}$. This yields $5d = z^2$. So $z = 5z_0$ for some integer z_0. We get $d = 5z_0^2$. Since d is a square free integer, we have $d = 5$. It follows that $a = 0$ and $b = -1$ or $a = -1$ and $b = 1$. Therefore, $x = \frac{-1\mp\sqrt{5}}{2}, y = \frac{-1\pm\sqrt{5}}{2}$.

Case 2.4 $N(x) = N(x-1) = -1$. Hence $b = 1-2a$ and $N(x) = a^2+a(1-2a)+(1-2a)^2((1-d)/4) = -1$. This implies that $a = \frac{1-4M\pm\sqrt{5-20M}}{2(1-4M)}$ where $M = (1-d)/4$. Thus $5-20M = z^2$ for some $z \in \mathbb{Z}$. This yields $5d = z^2$. So $z = 5z_0$ for some integer z_0. We get $d = 5z_0^2$. Since d is a square free integer, $d = 5$. It follows that $a = 0$ and $b = 1$ or $a = 1$ and $b = -1$. Hence $x = \frac{1\pm\sqrt{5}}{2}, y = \frac{3\pm\sqrt{5}}{2}$. □

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