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Upper and lower (τ_1, τ_2)-continuous multifunctions

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Abstract

In this paper, we introduce the notions of upper and lower (τ_1, τ_2)-continuous multifunctions. We also investigate some characterizations of upper and lower (τ_1, τ_2)-continuous multifunctions.

1 Introduction

Continuity for functions is an important concept for the study and investigation in topological spaces. This concept has been extended to the setting of multifunctions and has been generalized by weaker forms of open sets. Several different forms of continuous multifunctions have been introduced and

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studied over the years. In 1996, Popa and Noiri [11] obtained some characterizations of upper and lower α-continuous multifunctions. Moreover, Popa and Noiri [10] introduced and studied the notions of upper and lower β-continuous multifunctions. In 2000, Noiri and Popa [7] investigated the concepts upper and lower M-continuous multifunctions as multifunctions defined between sets satisfying certain minimal conditions. Popa and Noiri [9] introduced and investigated the notion of m-continuous multifunctions. In 2004, Park et al. [8] introduced and studied δ-precontinuous multifunctions as a generalization of precontinuous multifunctions due to Popa [12]. Laprom et al. [6] introduced and studied the notions of upper and lower $\beta(\tau_1, \tau_2)$-continuous multifunctions. Viriyapong and Boonpok [13] introduced and investigated the concepts of upper and lower $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Moreover, several characterizations of $(\tau_1, \tau_2)\delta$-semicontinuous mutifunctions and almost weakly (τ_1, τ_2)-continuous multifunctions were established in [4] and [3], respectively. In this paper, we introduce the notions of upper and lower (τ_1, τ_2)-continuous multifunctions and discuss some characterizations of upper and lower (τ_1, τ_2)-continuous multifunctions.

2 Preliminaries

Throughout this paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [5] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [5] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [5] of A and is denoted by $\tau_1\tau_2$-Int(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [13] (resp. $(\tau_1, \tau_2)s$-*open* [4]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each

$A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3 Upper and lower (τ_1, τ_2)-continuous multifunctions

In this section, we introduce the notions of upper and lower (τ_1, τ_2)-continuous multifunctions. Moreover, we investigate some characterizations of upper and lower (τ_1, τ_2)-continuous multifunctions.

Definition 3.1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:*

(1) upper (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$;

(2) lower (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$.

Theorem 3.2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper (τ_1, τ_2)-continuous;

(2) $F^+(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^-(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-Cl$(F^-(B)) \subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y;

(5) $F^+(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-Int$(F^+(B))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then $F(x) \subseteq V$ and by (1), there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$. Thus, $x \in U \subseteq F^+(V)$ and hence $x \in \tau_1\tau_2$-Int$(F^+(V))$. Therefore, $F^+(V) \subseteq \tau_1\tau_2$-Int$(F^+(V))$. This shows that $F^+(V)$ is $\tau_1\tau_2$-open in X.

(2) $\Rightarrow$ (3): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2$-Cl(B) is $\sigma_1\sigma_2$-closed in Y and by (3), $\tau_1\tau_2$-Cl$(F^-(B)) \subseteq \tau_1\tau_2$-Cl$(F^-(\sigma_1\sigma_2$-Cl$(B))) = F^-(\sigma_1\sigma_2$-Cl$(B))$

(4) $\Rightarrow$ (5): Let B be any subset of Y. By (4), we have

$$\begin{aligned} X - \tau_1\tau_2\text{-Int}(F^+(B)) &= \tau_1\tau_2\text{-Cl}(X - F^+(B)) \\ &= \tau_1\tau_2\text{-Cl}(F^-(Y - B)) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &= F^-(Y - \sigma_1\sigma_2\text{-Int}(B)) \\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(B)) \end{aligned}$$

and hence $F^+(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_1\tau_2\text{-Int}(F^+(B))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. Then $x \in F^+(V) = \tau_1\tau_2\text{-Int}(F^+(V))$. There exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq F^+(V)$; hence $F(U) \subseteq V$. This shows that F is upper (τ_1, τ_2)-continuous. □

Theorem 3.3. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower (τ_1, τ_2)-continuous;

(2) $F^-(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^+(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2\text{-}Cl(F^+(B)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $F(\tau_1\tau_2\text{-}Cl(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(F(A))$ for every subset A of X;

(6) $F^-(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(F^-(B))$ for every subset B of Y.

Proof. We only prove the implications (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (6) with the other proofs being similar to those of Theorem 3.2.

(4) $\Rightarrow$ (5): Let A be any subset of X. By (4),

$$\tau_1\tau_2\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(F^+(F(A))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(F(A)))$$

and hence $F(\tau_1\tau_2\text{-Cl}(A)) \subseteq \sigma_1\sigma_2\text{-Cl}(F(A))$.

(5) $\Rightarrow$ (6): Let B be any subset of Y. By (5),

$$\begin{aligned} F(\tau_1\tau_2\text{-Cl}(F^+(Y - B))) &\subseteq \sigma_1\sigma_2\text{-Cl}(F(F^+(Y - B))) \\ &\subseteq \sigma_1\sigma_2\text{-Cl}(Y - B) \\ &= Y - \sigma_1\sigma_2\text{-Int}(B). \end{aligned}$$

Since

$$F(\tau_1\tau_2\text{-Cl}(F^+(Y - B))) = F(\tau_1\tau_2\text{-Cl}(X - F^-(B))) \\ = F(X - \tau_1\tau_2\text{-Int}(F^-(B))),$$

we have $X - \tau_1\tau_2\text{-Int}(F^-(B)) \subseteq F^+(Y - \sigma_1\sigma_2\text{-Int}(B)) = X - F^-(\sigma_1\sigma_2\text{-Int}(B))$ and hence $F^-(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_1\tau_2\text{-Int}(F^-(B))$. □

Definition 3.4. [2] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous if f has this property at each point of X.*

Corollary 3.5. [2] *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is (τ_1, τ_2)-continuous;

(2) $f^{-1}(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $f(\tau_1\tau_2\text{-}Cl(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(f(A))$ for every subset A of X;

(4) $\tau_1\tau_2\text{-}Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(B))$ for every subset B of Y;

(6) $f^{-1}(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-closed set K of Y.

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