International Journal of Mathematics and
Computer Science, **19**(2024), no. 4, 1261–1264

Independence and Star Polynomials: Interrelations in Biclique Polynomials

Shaleema A. Arriola[1], **Benjier H. Arriola**[1],
Regimar A. Rasid[2], **Jamaluddin S. Sala**[2]

[1]Mathematics Department
College of Education
Basilan State College, Sumagdang, Isabela City
7300 Basilan, Philippines

[2]Mindanao State University
Tawi-Tawi College of Technology and Oceanography
7500 Bongao, Tawi-Tawi, Philippines

email: shallarriola1023@yahoo.com, benj_arriola@yahoo.com,
regimarrasid@msutawi-tawi.edu.ph, jamaluddinsala@msutawi-tawi.edu.ph

(Received April 17, 2024, Accepted May 23, 2024,
Published June 1, 2024)

Abstract

In this paper, we establish the biclique polynomial of graphs resulting from the corona of two connected graphs in terms of the independence polynomials and star polynomials.

1 Introduction

The study of representing a graph in terms of a polynomial garnered interests recently since this representation captured applications in other fields of sciences such as Chemistry, Biology, and Physics [3]. Several discrete mathematicians established many results by considering specific subgraph structures of a given graph. Hoede and Li [4] investigated the independent

Key words and phrases: biclique, independent set, star polynomial, biclique polynomial.
AMS (MOS) Subject Classifications: 05C25, 05C30, 05C31.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

set polynomial of graphs which counts the number of independent subsets of the vertex-set of a graph of all possible cardinalities. Villarta et al. [7] established the induced path polynomials in the join and the corona of graphs. Moreover, Maldo and Artes [6] pioneered a work on geodetic independence polynomial of a graph and established a result using the Chuh-Shih-Chieh's Identity. The biclique polynomial was first introduced by Lumpayao et al. [5]. This polynomial counts the number of bicliques in a graph of all orders.

2 Interrelations

It is interesting to note that stars are bicliques and the partite sets of bicliques are independent sets. These facts give us ideas on the structure of graphs resulting from the corona of two graphs and how to establish the biclique polynomial in terms of star polynomial and independent polynomial of graphs being considered in the operation.

A biclique in G is a subset of $V(G)$ which induces a complete bipartite graph in G. The *biclique polynomial* of a graph G, denoted by $\Gamma_b(G;x)$, is given by $\Gamma_b(G;x) = \sum_{i=2}^{\beta(G)} b_i(G)x^i$, where $b_i(G)$ is the number of bicliques in G of cardinality i and $\beta(G)$ is the cardinality of the maximum biclique in G [5].

A subset S of $V(G)$ is an *independent set* in G if the elements of S are pairwise non-adjacent. The *independence polynomial* of G is given by $I(G;x) = \sum_{i=1}^{\alpha(G)} \alpha_i(G)x^i$, where $\alpha_i(G)$ is the number of independent subsets of $V(G)$ of cardinality i and $\alpha(G)$ is the independence number of G, the maximum cardinality of an independent set in G [4].

The star polynomial of a graph G is defined as $\Gamma_s(G;x) = \sum_{i=1}^{s(G)} s_i(G)x^i$, where $s_i(G)$ is the number of induced starts in G of cardinality i and $s(G)$ is the cardinality of the maximum induced star in G. Artes Jr. et al. established results on special graphs and the corona of graphs ([1],[2]).

First, we characterize the bicliques in the corona of graphs. Given two connected graphs G and H, the *corona* $G \circ H$ of G with H has vertex-set $V(G \circ H) = V(G) \cup \bigcup_{v \in V(G)} V(H_v)$, where H_v is a copy of H attached to

$v \in V(G)$, and edge-set

$$E(G \circ H) = E(G) \cup \bigcup_{v \in V(G)} \left[E(H_v) \cup \{uz : u \in V(G), z \in V(H_v)\} \right].$$

The adjacency in $G \circ H$ carries the adjacency of G and the adjacency in H in every copy H_v of H and adding additional edges by joining each vertex $v \in V(G)$ to every vertex of H_v.

The bicliques in the corona of graphs are characterized in the following lemma.

Lemma 2.1. *Suppose G and H are nontrivial connected graphs. A subset S of $V(G \circ H)$ is a biclique in $G \circ H$ if and only if it satisfies one of the following conditions:*

(i) *S is a biclique in G*

(ii) *S is a biclique in a copy of H*

(iii) *$S = S_G \cup S_H$ where S_G induces a star in G and S_H is an independent set in H.*

Proof. Assume $S \subseteq V(G \circ H)$ is a biclique in $G \circ H$. Then $S = S_1 \cup S_2$, where S_1 and S_2 are the partite sets. We have the following cases:

Case 1: $S \cap V(G) \neq \varnothing$.

Subcase 1.1: $S \cap V(H_v) = \varnothing$ for every copy H_v of H attached to $v \in V(G)$.

In this case, the set S is a biclique in G and (i) is satisfied.

Subcase 1.2: There exists a vertex in G satisfying $S \cap V(H_v) \neq \varnothing$.

Then $v \in S$. Note that if $u \in V(G)$ and u is different from v, then $S \cap V(H_u) = \varnothing$ by biclique properties. Now, the only vertex in G adjacent to H_v is the vertex v. This forces $V(H_v) \cap S$ to be independent in H_v and $S \cap V(G)$ must induce a star $K_{1,|S \cap V(G)|-1}$ in G which is also a star in $G \circ H$. Hence (iii) is satisfied.

Case 2: $S \cap V(G) = \varnothing$.

Note that S can only intersect one copy of H; say, H_v, where $v \in V(G)$. Thus S is a biclique in a copy of H and hence a biclique of H. Thus (ii) is satisfied.

Conversely, any conditions in the lemma will imply that S is a biclique in $G \circ H$. $\square$

Finally, we have the following theorem which establishes the biclique polynomial of the corona.

Theorem 2.2. *Suppose that G and H are nontrivial connected graphs. Then*

$$\Gamma_b(G \circ H; x) \;=\; \Gamma_b(G; x) + |V(G)|\Gamma_b(H; x) + \Gamma_s(G; x)I(H; x).$$

Proof. Let S be a biclique in $G \circ H$. Then S satisfies the conditions of Lemma 2.1. The first condition gives us the first term of the polynomial. The second condition gives us the second term of the polynomial. The last term of the polynomial follows from condition (*iii*) of Lemma 2.1. $\square$

The independence polynomial and star polynomial play significant roles in establishing our main result.

References

[1] R.G. Artes Jr., N.H.R. Mohammad, A.A. Laja, N.H.M. Hassan, From Graphs to Polynomial Rings: Star Polynomial Representation of Graphs, *Advances and Applications in Discrete Mathematics*, **37,** (2023), 67–76. https://doi.org/10.17654/0974165823012

[2] R.G. Artes Jr., N.H.R. Mohammad, Z.H. Dael, H.B. Copel, Star Polynomial of the Corona of Graphs. *Advances and Applications in Discrete Mathematics*, **39,** no. 1, (2023), 81–87. https://doi.org/10.17654/0974165823037

[3] J. Ellis-Monaghan, J. Merino, *Graph Polynomials and Their Applications II: Interrelations and Interpretations*, Birkhauser, Boston, 2011.

[4] C. Hoede, X. Li, Clique polynomials and independent set polynomials of graphs, *Discrete Mathematics*, **125,** (1994), 219–228.

[5] S.M. Lumpayao, R.A. Rasid, R.G. Artes Jr., On Biclique Polynomials. *Advances and Applications in Discrete Mathematics*, **41,** no. 3, (2024), 231–237. https://doi.org/10.17654/0974165824017

[6] J.F.B. Maldo, R.G. Artes Jr., Applications of Chuh-Shih-Chieh's Identity in Geodetic Independence Polynomials, *International Journal of Mathematics and Computer Science*, **19,** no. 3, (2024), 642–647.

[7] C.A. Villarta, R.G. Eballe, R.G. Artes Jr., Induced Path Polynomials of the Join and Corona of Graphs, *International Journal of Mathematics and Computer Science*, **19,** no. 3, (2024), 649–652.