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# Uncertainty Principle in Reproducing Kernel Hilbert Spaces

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**Abstract**

The Reproducing Kernel Hilbert space is a special class of Hilbert spaces in which the evaluation functional is continuous and bounded. In this paper, we investigate the Uncertainty Principle in some Reproducing Kernel Hilbert.

## 1 Introduction

The uncertainty principle is one of the celebrated principles in mathematics and has encompassed other fields of study, particularly in quantum mechanics. Over the years, mathematicians had delved in this principle and have developed their own variations according to their domain. The mathematical

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survey by Folland and Sitaram [2] had summed it up as follows: a nonzero function and its Fourier transform cannot both be sharply localized.

In quantum mechanics, the uncertainty principle states that some parameters cannot be measured simultaneously without endangering the accuracy of the other. An example of this is the position and momentum of a particle. This has led physicists to abandon the classical Newtonian laws of mechanics in dealing with states at a quantum level [1]. In a classical experiment, a system can be represented as a point in a plane called the classical phase space and every parameter related to its movement can be computed through its coordinates. This straight-forward approach is null in the quantum level, and thus mathematicians and physicists embarked on the quest of finding the best suited representation of a quantum system. They found out that a state of a quantum system can be represented as a unit vector in a complex Hilbert space, and an observable, or a quantity to be measured, as a Hermitian operator [3].

In [6], the following are defined: Given $f \in L^2(\mathbb{R})$ with $\|f\|_{L^2(\mathbb{R})} = 1$, we define its associated mean

$$\mu(f) = \text{Mean}(|f|^2) = \int t|f(t)|^2 dt,$$

and variance

$$\Delta^2(f) = \text{Var}(|f|^2) = \int |t - \mu(f)|^2 |f(t)|^2 dt.$$

## 2 The Space $\mathcal{H}_\varphi(\mathbb{R})$

Fix $\varphi \in L^2(\mathbb{R})$ such that $\hat{\varphi} \geq 0$ on $\mathbb{R}$ and $\hat{\varphi} \in L^1(\mathbb{R})$. Define

$$\Sigma(\hat{\varphi}) = \{t \in \mathbb{R} : \hat{\varphi}(t) \neq 0\}.$$

Define $\mathcal{H}_\varphi(\mathbb{R})$ as the vector space of all $f \in L^2(\mathbb{R})$ such that $\hat{f}(\omega) = 0$ if $\hat{\varphi}(\omega) = 0$ and

$$\int_{\Sigma(\hat{\varphi})} \frac{|\hat{f}(\omega)|^2}{\hat{\varphi}(\omega)} d\omega < \infty.$$

Note that $\varphi \in \mathcal{H}_\varphi(\mathbb{R})$. Indeed,

$$\int_{\Sigma(\hat{\varphi})} \frac{|\hat{\varphi}(\omega)|^2}{\hat{\varphi}(\omega)} d\omega = \int_{\Sigma(\hat{\varphi})} \hat{\varphi}(\omega) d\omega = \int_{\mathbb{R}} \hat{\varphi}(\omega) d\omega < \infty,$$

since $\hat{\varphi} \in L^1(\mathbb{R})$. For $f, g \in \mathcal{H}_\varphi(\mathbb{R})$, define

$$\langle f, g\rangle_{\mathcal{H}_\varphi(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \int_{\Sigma(\hat{\varphi})} \frac{\hat{f}(\omega)\overline{\hat{g}(\omega)}}{\hat{\varphi}(\omega)} d\omega$$

and

$$\|f\|^2_{\mathcal{H}_\varphi(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \int_{\Sigma(\hat{\varphi})} \frac{|\hat{f}(\omega)|^2}{\hat{\varphi}(\omega)} d\omega$$

**Lemma 2.1.** *Let $\varphi \in L^2(\mathbb{R})$. Fix $x \in \mathbb{R}$ and define $\varphi_x(t) = \varphi(t - x)$ for each $t \in \mathbb{R}$. Then*

*(a) $\hat{\varphi}_x(\omega) = e^{-i\omega x}\hat{\varphi}(\omega)$, for almost every $\omega \in \mathbb{R}$*

*(b) If $\varphi$ is $k$-times differentiable for $k \in \mathbb{N}$, then $\varphi_x$ is also $k$-times differentiable. Moreover, $\hat{\varphi_x^{(k)}}(\omega) = e^{-i\omega x}\hat{\varphi^{(k)}}(\omega)$, for almost every $\omega \in \mathbb{R}$*

*(c) $\varphi_x \in \mathcal{H}_\varphi(\mathbb{R})$*

*Proof.* Fix $x \in \mathbb{R}$ and $\varphi \in L^2(\mathbb{R})$. Then

$$\begin{aligned} \hat{\varphi}_x(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi_x(t) e^{-i\omega t} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(t - x) e^{-i\omega t} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(s) e^{-i\omega(s+x)} ds \\ &= e^{-i\omega x}\hat{\varphi}(\omega) \end{aligned}$$

for almost every $\omega \in \mathbb{R}$. For the second part, fix $k \in \mathbb{N}$. Then

$$\begin{aligned} \hat{\varphi_x}^{(k)}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi_x^{(k)} e^{-i\omega t} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi^{(k)}(t - x) e^{-i\omega t} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi^{(k)}(s) e^{-i\omega(s+x)} ds \\ &= e^{-i\omega x}\hat{\varphi^{(k)}}(\omega) \end{aligned}$$

for almost every $\omega \in \mathbb{R}$. Finally, note that from the first part $\hat{\varphi_x}(\omega) = e^{-i\omega x}\hat{\varphi}(\omega) = 0$ whenever $\hat{\varphi}(\omega) = 0$. Using the first part, we have

$$\begin{aligned} \int_{\Sigma(\hat{\varphi})} \frac{|\hat{\varphi_x}(\omega)|^2}{\hat{\varphi}(\omega)} d\omega &= \int_{\Sigma(\hat{\varphi})} \frac{|e^{-i\omega x}\hat{\varphi}(\omega)|^2}{\hat{\varphi}(\omega)} d\omega \\ &= \int_{\Sigma(\hat{\varphi})} \hat{\varphi}(\omega) d\omega \\ &= \int_{\mathbb{R}} \hat{\varphi}(\omega) d\omega < \infty \end{aligned}$$

since $\hat{\varphi} \in L^1(\mathbb{R})$. It follows that $\varphi_x \in \mathcal{H}_\varphi(\mathbb{R})$. □

**Proposition 2.2.** $\mathcal{H}_\varphi(\mathbb{R})$ *has the following properties:*

*(a)* $\mathcal{H}_\varphi(\mathbb{R})$ *is a Hilbert space*

*(b) If* $f \in \mathcal{H}_\varphi(\mathbb{R})$*, then* $\hat{f} \in L^1(\mathbb{R})$

*(c) For each* $x \in \mathbb{R}$*,* $\langle f, \varphi_x \rangle_{\mathcal{H}_\varphi(\mathbb{R})} = f(x)$ *for every* $f \in \mathcal{H}_\varphi(\mathbb{R})$*.*

*Proof.* Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{H}_\varphi(\mathbb{R})$. Now for $m, n \in \mathbb{N}$,

$$\begin{aligned} \|f_n - f_m\|^2_{\mathcal{H}_\varphi(\mathbb{R})} &= \frac{1}{\sqrt{2\pi}} \int_{\Sigma(\hat{\varphi})} \frac{|\hat{f}_n(\omega) - \hat{f}_m(\omega)|^2}{\hat{\varphi}(\omega)} d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{\Sigma(\hat{\varphi})} |\frac{\hat{f}_n(\omega)}{\sqrt{\hat{\varphi}(\omega)}} - \frac{\hat{f}_m(\omega)}{\sqrt{\hat{\varphi}(\omega)}}|^2 d\omega. \end{aligned}$$

Hence $\left\{\frac{\hat{f}_n}{\sqrt{\hat{\varphi}}}\right\}_{n=1}^{\infty}$ is also a Cauchy sequence in $\mathcal{H}_\varphi(\mathbb{R})$ for which $\hat{\varphi}$ does not vanish and hence $\left\{\frac{\hat{f}_n}{\sqrt{\hat{\varphi}}}\right\}_{n=1}^{\infty}$ is also a Cauchy sequence in $L^2(\mathbb{R})$. Thus there exists $h \in L^2(\mathbb{R})$ such that $\left\|\frac{\hat{f}_n}{\sqrt{\hat{\varphi}}} - h\right\|_{L^2(\mathbb{R})} \to 0$ as $n \to \infty$.

Note that $g = h\sqrt{\hat{\varphi}} \in L^2(\mathbb{R})$. Indeed, by Cauchy-Schwarz Inequality,

$$\int_{\mathbb{R}} |g(x)|^2 dx = \int_{\mathbb{R}} |h(x)\sqrt{\hat{\varphi}(x)}|^2 dx = \int_{\mathbb{R}} |h(x)|^2 \hat{\varphi}(x) dx \le \|\hat{\varphi}\|_\infty \|h\|^2_{L^2(\mathbb{R})}$$

where $\| \cdot \|$ denotes the uniform norm.

Since the Fourier transform is a Hilbert space isomorphism from $L^2(\mathbb{R})$ onto itself, there exists $f \in L^2(\mathbb{R})$ such that $\hat{f} = g$. Note that $\hat{f}(\omega) = g(\omega) = h(\omega)\sqrt{\hat{\varphi}(\omega)} = 0$ whenever $\hat{\varphi}(\omega) = 0$. Also, $\dfrac{\hat{f}(\omega)}{\sqrt{\hat{\varphi}(\omega)}} = \dfrac{g(\omega)}{\sqrt{\hat{\varphi}(\omega)}} = h(\omega)$. Hence $\displaystyle\int_{\Sigma(\hat{\varphi})} \frac{|\hat{f}(o)|^2}{\hat{\varphi}(\omega)} d\omega = \int_{\Sigma(\hat{\varphi})} \left| \frac{\hat{f}(\omega)}{\sqrt{\hat{\varphi}(\omega)}} \right|^2 d\omega = \int_{\Sigma(\hat{\varphi})} |h(\omega)|^2 d\omega < \infty$.

Thus $f \in \mathcal{H}_\varphi(\mathbb{R})$ and

$$\begin{aligned} \|f_n - f\|^2_{\mathcal{H}_\varphi(\mathbb{R})} &= \frac{1}{\sqrt{2\pi}} \int_{\Sigma(\hat{\varphi})} \left| \frac{\hat{f}_n(\omega)}{\sqrt{\hat{\varphi(\omega)}}} - \frac{\hat{f}(\omega)}{\sqrt{\hat{\varphi}(\omega)}} \right|^2 d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{\Sigma(\hat{\varphi})} \left| \frac{\hat{f}_n(\omega)}{\sqrt{\hat{\varphi}(\omega)}} - h(\omega) \right|^2 d\omega \to 0 \end{aligned}$$

as $n \to \infty$ by the pointwise convergence in $L^2(\mathbb{R})$. This proves the first part.

For the second part, fix $f \in \mathcal{H}_\varphi(\mathbb{R})$. Then $\hat{f} \equiv 0$ outside $\Sigma(\hat{\varphi})$ and by Cauchy-Schwarz Inequality we have

$$\begin{aligned} \int_{\mathbb{R}} |\hat{f}(\omega)| d\omega &= \int_{\Sigma(\hat{\varphi})} |\hat{f}(\omega)| d\omega + \int_{\mathbb{R}-\Sigma(\hat{\varphi})} |\hat{f}(\omega)| d\omega \\ &= \int_{\Sigma(\hat{\varphi})} \frac{|\hat{f}(\omega)|}{\sqrt{\hat{\varphi}(\omega)}} \sqrt{\hat{\varphi}(\omega)} d\omega \\ &\leq \left( \int_{\Sigma(\hat{\varphi})} \frac{|\hat{f}(\omega)|^2}{\hat{\varphi}(\omega)} d\omega \right)^{1/2} \left( \int_{\Sigma(\hat{\varphi})} \hat{\varphi}(\omega) d\omega \right)^{1/2} < \infty. \end{aligned}$$

Accordingly, $\hat{f} \in L^1(\mathbb{R})$.

Lastly, fix $x \in \mathbb{R}$ and $f \in \mathcal{H}_\varphi(\mathbb{R})$. Since $\hat{\varphi} \geq 0$, Lemma 2.1 implies that $\overline{\hat{\varphi}_x(\omega)} = e^{i\omega x}\hat{\varphi}(\omega)$ for almost every $\omega \in \Sigma(\hat{\varphi})$. Thus we have

$$\langle f, \varphi_x \rangle_{\mathcal{H}_\varphi(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \int_{\Sigma(\hat{\varphi})} \frac{\hat{f}(\omega)\overline{\hat{\varphi}_x(\omega)}}{\hat{\varphi}(\omega)} d\omega = \frac{1}{\sqrt{2\pi}} \int_{\Sigma(\hat{\varphi})} \hat{f}(\omega) e^{i\omega x} d\omega = f(x).$$

# 3 Main Result

We have the analogous result for the odd kernel and its transform in $\mathcal{H}_\varphi(\mathbb{R})$.

**Theorem 3.1.** *Let $\varphi \in L^2(\mathbb{R})$ be an odd function. Then for every $t \in \mathbb{R}$,*

$$\Delta^2(\varphi_t)\Delta^2(\hat{\varphi}_t) \geq (\|\varphi\|^2_{L^2(\mathbb{R})} + |t|^2\|\varphi\|^6_{L^2(\mathbb{R})})\|\hat{\varphi}\|^2_{L^2(\mathbb{R})}.$$

*Proof.* Let $\varphi \in L^2(\mathbb{R})$ be odd and $t \in \mathbb{R}$. Then

$$\begin{aligned} \mu(\varphi_t) &= \int x|\varphi_t(x)|^2 dx \\ &= \int x|\varphi(x-t)|^2 dx \\ &= \int (s+t)|\varphi(x)|^2 ds, \qquad \text{where } s = x-t \\ &= \int s|\varphi(s)|^2 ds + \int t|\varphi(s)|^2 ds \\ &= \mu(\varphi) + t\|\varphi\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Moreover,

$$\mu(\hat{\varphi}_t) = \int \omega|\hat{\varphi}_t(\omega)|^2 d\omega = \int \omega|e^{-i\omega t}\hat{\varphi}(\omega)|^2 d\omega = \int \omega|\hat{\varphi}(\omega)|^2 d\omega = \mu(\hat{\varphi}).$$

Then

$$\begin{aligned} \Delta^2(\varphi_t) &= \int |x-\mu(\varphi_t)|^2|\varphi_t(x)|^2 dx \\ &= \int |x-\mu(\varphi_t)|^2|\varphi(x-t)|^2 dx \\ &= \int |s+t-(\mu(\varphi)+t\|\varphi\|^2_{L^2(\mathbb{R})})|^2|\varphi(s)|^2 ds \qquad \text{where } s = x-t \\ &= \int |s+t-t\|\varphi\|^2_{L^2(\mathbb{R})}|^2|\varphi(s)|^2 ds \qquad \text{since } \varphi \text{ is odd} \\ &= \int |s-t(\|\varphi\|^2_{L^2(\mathbb{R})}-1)|^2|\varphi(s)|^2 ds \\ &\geq \int (|s|^2-|t(\|\varphi\|^2_{L^2(\mathbb{R})}-1)|^2)|\varphi(s)|^2 ds \\ &\geq \int |s|^2|\varphi(s)|^2 ds - \int |t|^2(\|\varphi\|^2_{L^2(\mathbb{R})}-1)^2|\varphi(s)|^2 ds \\ &\geq \int |s||\varphi(s)|^2 ds - |t|^2(\|\varphi\|^2_{L^2(\mathbb{R})}-1)^2(\|\varphi\|^2_{L^2(\mathbb{R})}) \\ &= \mu(\varphi) - t^2\|\varphi\|^2_{L^2(\mathbb{R})}(\|\varphi\|^2_{L^2(\mathbb{R})}-1)^2 \end{aligned}$$

and

$$\begin{aligned}
\Delta^2(\hat{\varphi}_t) &= \int |x-\mu(\hat{\varphi}_t)|^2|\hat{\varphi}_t(x)|^2dx \\
&= \int |x-\mu(\hat{\varphi})|^2|e^{-ixt}\hat{\varphi}(x)|^2dx \\
&= \int |x-\mu(\hat{\varphi})|^2|\hat{\varphi}(x)|^2dx \\
&\geq \int (|x|^2-|\mu(\hat{\varphi})|^2)|\hat{\varphi}(x)|^2dx \\
&= \int |x|^2|\hat{\varphi}(x)|^2dx - \int |\mu(\hat{\varphi})|^2|\hat{\varphi}(x)|^2dx \\
&\geq \int |x||\hat{\varphi}(x)|^2dx - (\mu(\varphi))^2\|\hat{\varphi}\|^2_{L^2(\mathbb{R})} \\
&= \mu(\hat{\varphi}) - \mu(\hat{\varphi})\|\hat{\varphi}\|^2_{L^2(\mathbb{R})} \\
&= \mu(\hat{\varphi})(1-\|\hat{\varphi}\|^2_{L^2(\mathbb{R})})
\end{aligned}$$

Combining gives the desired result. □

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