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Some identities of (s, t)-Pell and (s, t)-Pell-Lucas polynomials by matrix methods

Somnuk Srisawat, Wanna Sripad

Department of Mathematics and Computer Science
Faculty of Science and Technology
Rajamangala University of Technology Thanyaburi
Pathum Thani 12110, Thailand

email: somnuk_s@rmutt.ac.th
wanna_sriprad@rmutt.ac.th

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Abstract

In this paper, we explore the extensions of Pell and Pell-Lucas polynomials known as (s, t)-Pell and (s, t)-Pell-Lucas polynomials. We introduce the 2×2 matrices denoted as A and B to facilitate our investigation. By using these matrices, we derive various identities and summation formulas for (s, t)-Pell and (s, t)-Pell-Lucas polynomials.

1 Introduction

For several years, many researchers have extensively investigated numerous polynomial sequences. The most famous polynomials are Fibonacci, Lucas, Pell, and Pell-Lucas polynomials which are also renowned for their diverse range of remarkable properties and wide-ranging applications in mathematics, physics, and computer science [1, 2]. These polynomials have garnered

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substantial interest. Various authors have recently generalized and studied the Fibonacci, Lucas, Pell, and Pell-Lucas polynomials, as evidenced by works such as [3, 4, 5, 6].

In 2021, Srisawat and Sriprad [6] introduced a generalization of the Pell and Pell-Lucas polynomials as follows:
Let x be a real variable and let s and t be real numbers with $s^2x^2+t>0$, $s>0$, and $t\neq 0$. The (s,t)-Pell polynomials $\{P_n(s,t)(x)\}_{n=0}^{\infty}$ and the (s,t)-Pell-Lucas polynomial sequences $\{Q_n(s,t)(x)\}_{n=0}^{\infty}$ are defined respectively by

$$P_n(s,t)(x) = 2sxP_{n-1}(s,t)(x) + tP_{n-2}(s,t)(x), \quad \text{for } n \geq 2, \tag{1.1}$$

$$Q_n(s,t)(x) = 2sxQ_{n-1}(s,t)(x) + tQ_{n-2}(s,t)(x), \quad \text{for } n \geq 2, \tag{1.2}$$

with the initial conditions $P_0(s,t)(x)=0$, $P_1(s,t)(x)=1$, and $Q_0(s,t)(x)=2$, $Q_1(s,t)(x)=2sx$. Notably, when $s=\frac{1}{2}$ and $t=1$, these sequences yield the classical Fibonacci and Lucas polynomial sequences, while setting $s=t=1$ the classical Pell and Pell-Lucas polynomial sequences are obtained. The characteristic equation of (1.1) and (1.2) are in the form $\lambda^2=2sx\lambda+t$, and the roots of this equation are $\alpha=sx+\sqrt{s^2x^2+t}$ and $\beta=sx-\sqrt{s^2x^2+t}$. We note that $\alpha+\beta=2sx, \alpha-\beta=2\sqrt{s^2x^2+t}$ and $\alpha\beta=-t$. Srisawat and Sriprad [6] also gave the Binet's formulas and the generating functions for these polynomials and obtained many identities of these polynomials by using the Binet's formula. We have [6] $P_{n+1}(s,t)(x)+tP_{n-1}(s,t)(x)=Q_n(s,t)(x), 2sxP_n(s,t)(x)+2tP_{n-1}(s,t)(x)=Q_n(s,t)(x)$, and $Q_{n+1}(s,t)(x)+tQ_{n-1}(s,t)(x)=4(s^2x^2+t)P_n(s,t)(x)$, for all $n\in\mathbb{N}$.

In this paper, we determine the 2×2 A-matrix and B-matrix. After that, we establish some identities of (s,t)-Pell and (s,t)-Pell-Lucas polynomials by using these matrices. For convenience and throughout this paper, we use the symbols $\mathcal{P}_n(x)$ and $\mathcal{Q}_n(x)$ instead of $P_n(s,t)(x)$ and $Q_n(s,t)(x)$, respectively.

2 Main results

In this section, we establish some identities of (s,t)-Pell and (s,t)-Pell-Lucas polynomials by using the 2×2 A-matrix and B-matrix define as follows:

$$A = \begin{bmatrix} 2sx & t \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2sx & 2t \\ 2 & -2sx \end{bmatrix}, \tag{2.3}$$

where s,x,t are defined as in (1.1) and (1.2).

By using the A-matrix, we get the following lemma.

Lemma 2.1. *Let* A *be a matrix as in* (2.3)*. Then*

$$A^n = \begin{bmatrix} \mathcal{P}_{n+1}(x) & t\mathcal{P}_n(x) \\ \mathcal{P}_n(x) & t\mathcal{P}_{n-1}(x) \end{bmatrix}, \text{ for all } n \in \mathbb{N}.$$

Proof. The proofs follow by induction on n. □

Next, by using the A-matrix and B-matrix, we get the following lemma:

Lemma 2.2. *Let* A, B *be matrices as in* (2.3)*. Then*

$$A^n B = BA^n = \begin{bmatrix} \mathcal{Q}_{n+1}(x) & t\mathcal{Q}_n(x) \\ \mathcal{Q}_n(x) & t\mathcal{Q}_{n-1}(x) \end{bmatrix}, \text{ for all } n \in \mathbb{N}.$$

Proof. By Lemma 2.1, we have $A^n = \begin{bmatrix} \mathcal{P}_{n+1}(x) & t\mathcal{P}_n(x) \\ \mathcal{P}_n(x) & t\mathcal{P}_{n-1}(x) \end{bmatrix}$. Clearly,

$$A^n B = BA^n = \begin{bmatrix} \mathcal{Q}_{n+1}(x) & t\mathcal{Q}_n(x) \\ \mathcal{Q}_n(x) & t\mathcal{Q}_{n-1}(x) \end{bmatrix}.$$

□

In the following two theorems, we get the well-known Binet formulas and Cassini's identities by using the A-matrix, B-matrix, Lemma 2.1, and Lemma 2.2.

Theorem 2.3 (Binet's Formulas). *Let* $n \in \mathbb{N}$*. Then*

$$\mathcal{P}_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } \mathcal{Q}_n(x) = \alpha^n + \beta^n,$$

where α, β *are roots of the characteristic equation* $\lambda^2 - 2sx\lambda - t = 0$ *and* $\alpha > \beta$*.*

Proof. Let A, B be the matrices as in (2.3). Then the characteristic polynomial of A is $f(\lambda) = \det(A - \lambda I) = \lambda^2 - 2sx\lambda - t$, and the eigenvalues for A are $\alpha = sx + \sqrt{s^2x^2 + t}$ and $\beta = sx - \sqrt{s^2x^2 + t}$. Also, the eigenvectors for A corresponding to the eigenvalues α and β are $\begin{bmatrix} \alpha \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \beta \\ 1 \end{bmatrix}$, respectively. Let $W = \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix}$. Then $D = W^{-1}AW$ is a diagonal matrix. This implies that $A = WD^nW^{-1}$ and so $A^n = WD^nW^{-1}$. Then

$$A^n = \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha^{n+1} - \beta^{n+1} & t(\alpha^n - \beta^n) \\ \alpha^n - \beta^n & t(\alpha^{n-1} - \beta^{n-1}) \end{bmatrix}. \tag{2.4}$$

By Lemma 2.1, we have $A^n = \begin{bmatrix} \mathcal{P}_{n+1}(x) & t\mathcal{P}_n(x) \\ \mathcal{P}_n(x) & t\mathcal{P}_{n-1}(x) \end{bmatrix}$. Thus

$$\begin{bmatrix} \mathcal{P}_{n+1}(x) & t\mathcal{P}_n(x) \\ \mathcal{P}_n(x) & t\mathcal{P}_{n-1}(x) \end{bmatrix} = \frac{1}{\alpha-\beta}\begin{bmatrix} \alpha^{n+1}-\beta^{n+1} & t(\alpha^n-\beta^n) \\ \alpha^n-\beta^n & t(\alpha^{n-1}-\beta^{n-1}) \end{bmatrix}.$$

It follows that

$$\mathcal{P}_n(x) = \frac{\alpha^n-\beta^n}{\alpha-\beta}.$$

By (2.4), we have

$$A^nB = \frac{1}{\alpha-\beta}\begin{bmatrix} \alpha^{n+1}-\beta^{n+1} & t(\alpha^n-\beta^n) \\ \alpha^n-\beta^n & t(\alpha^{n-1}-\beta^{n-1}) \end{bmatrix}\begin{bmatrix} 2sx & 2t \\ 2 & -2sx \end{bmatrix}. \tag{2.5}$$

By (2.5) and Lemma 2.2, we obtain

$$\begin{bmatrix} \mathcal{Q}_{n+1}(x) & t\mathcal{Q}_n(x) \\ \mathcal{Q}_n(x) & t\mathcal{Q}_{n-1}(x) \end{bmatrix} = \begin{bmatrix} \alpha^{n+1}+\beta^{n+1} & t(\alpha^n+\beta^n) \\ \alpha^n+\beta^n & t(\alpha^{n-1}+\beta^{n-1}) \end{bmatrix}.$$

Consequently,

$$\mathcal{Q}_n(x) = \alpha^n+\beta^n.$$

□

Theorem 2.4 (Cassini's Identities). *Let $n \in \mathbb{N}$. Then*

(1) $\mathcal{P}_{n+1}(x)\mathcal{P}_{n-1}(x) - \mathcal{P}_n^2(x) = -(-t)^{n-1}$.

(2) $\mathcal{Q}_{n+1}(x)\mathcal{Q}_{n-1}(x) - \mathcal{Q}_n^2(x) = 4(s^2x^2+t)(-t)^{n-1}$.

Proof. Let A, B be the matrices as in (2.3). By Lemma 2.1 and the property $\det(A^n) = (\det(A))^n$, we get (1). In a similar way, by Lemma 2.2 and the property $\det(BA^n) = \det(B)(\det(A))^n$, we obtain (2). □

Furthermore, by using A-matrix, B-matrix, Lemma 2.1, and Lemma 2.2, we get some exciting identities, as in Theorems 2.5 and 2.7.

Theorem 2.5. *Let $m, n \in \mathbb{N}$. Then the following results hold:*

(1) $\mathcal{P}_{m+n}(x) = \mathcal{P}_m(x)\mathcal{P}_{n+1}(x) + t\mathcal{P}_{m-1}(x)\mathcal{P}_n(x)$.

(2) $\mathcal{Q}_{m+n}(x) = \mathcal{P}_{n+1}(x)\mathcal{Q}_m(x) + t\mathcal{P}_n(x)\mathcal{Q}_{m-1}(x)$.

(3) $(-t)^{n-1}\mathcal{P}_{m-n}(x) = \mathcal{P}_{m-1}(x)\mathcal{P}_n(x) - \mathcal{P}_m(x)\mathcal{P}_{n-1}(x)$.

(4) $(-t)^{n-1}\mathcal{Q}_{m-n}(x) = \mathcal{P}_n(x)\mathcal{Q}_{m-1}(x) - \mathcal{P}_{n-1}(x)\mathcal{Q}_m(x)$.

Proof. Let A, B be the matrices as in (2.3).
By Lemma 2.1 and the property that $A^{m+n} = A^m A^n$, we get (1). By Lemma 2.2 and the property that $BA^{m+n} = B(A^m A^n) = (BA^m)A^n$, we get (2). The proof of (3) and (4) go on in the same fashion as (1) and (2) by using the properties $A^{m-n} = A^m (A^n)^{-1}$ and $BA^{m-n} = B(A^m(A^n)^{-1}) = (BA^m)(A^n)^{-1}$. □

Putting $m = n$ and $m = n + 1$ in Theorem 2.5, we obtain the following corollary.

Corollary 2.6. *Let $n \in \mathbb{N}$. Then*

(1) $\mathcal{P}_{2n}(x) = \mathcal{P}_n(x)\mathcal{P}_{n+1}(x) + t\mathcal{P}_{n-1}(x)\mathcal{P}_n(x)$.

(2) $\mathcal{Q}_{2n}(x) = \mathcal{P}_{n+1}(x)\mathcal{Q}_n(x) + t\mathcal{P}_n(x)\mathcal{Q}_{n-1}(x)$.

(3) $\mathcal{P}_{2n+1}(x) = \mathcal{P}_{n+1}^2(x) + t\mathcal{P}_n^2(x)$.

(4) $\mathcal{Q}_{2n+1}(x) = \mathcal{P}_{n+1}(x)\mathcal{Q}_{n+1}(x) + t\mathcal{P}_n(x)\mathcal{Q}_n(x)$.

Theorem 2.7. *Let $n \in \mathbb{N}$ with $2sx + t - 1 \neq 0$. Then*

(1) $\sum_{k=0}^{n} \mathcal{P}_k(x) = \frac{\mathcal{P}_{n+1}(x) + t\mathcal{P}_n(x) - 1}{2sx+t-1}$,

(2) $\sum_{k=0}^{n} \mathcal{Q}_k(x) = \frac{\mathcal{Q}_{n+1}(x) + t\mathcal{Q}_n(x) - 2 + 2sx}{2sx+t-1}$.

Proof. Let A, B be the matrices as in (2.3). It is known that $I - A^{n+1} = (I - A)\sum_{k=0}^{n} A^k$. Since $\det(I - A) = 1 - 2sx - t \neq 0$, we can write

$$(I - A)^{-1}(I - A^{n+1}) = \sum_{k=0}^{n} A^k, \tag{2.6}$$

and we also obtain

$$(I - A)^{-1} = \frac{1}{1 - 2sx - t}\begin{bmatrix} 1 & t \\ 1 & 1 - 2sx \end{bmatrix}. \tag{2.7}$$

By Lemma 2.1, we get

$$I - A^{n+1} = \begin{bmatrix} 1 - \mathcal{P}_{n+2}(x) & -t\mathcal{P}_{n+1}(x) \\ -\mathcal{P}_{n+1}(x) & 1 - t\mathcal{P}_n(x) \end{bmatrix}. \tag{2.8}$$

From (2.7) and (2.8), we obtain

$$(I - A)^{-1}(I - A^{n+1}) = \begin{bmatrix} \frac{1 - \mathcal{P}_{n+2}(x) - t\mathcal{P}_{n+1}(x)}{1-2sx-t} & \frac{t - t\mathcal{P}_{n+1}(x) - t^2\mathcal{P}_n(x)}{1-2sx-t} \\ \frac{1 - \mathcal{P}_{n+2}(x) - (1-2sx)\mathcal{P}_{n+1}(x)}{1-2sx-t} & \frac{-t\mathcal{P}_{n+1}(x) + (1-2sx)(1-t\mathcal{P}_n(x))}{1-2sx-t} \end{bmatrix}. \tag{2.9}$$

From (2.6), (2.9) and Lemma 2.1, we get

$$\sum_{k=0}^{n} \mathcal{P}_k(x) = \frac{\mathcal{P}_{n+1}(x) + t\mathcal{P}_n(x) - 1}{2sx + t - 1}.$$

From (2.6), we also have

$$(I - A)^{-1}(I - A^{n+1})B = \sum_{k=0}^{n} A^k B. \tag{2.10}$$

Based on the same argument as above, result (2) is obtained by using (2.9), (2.10), and Lemma 2.2. This completes the proof. □

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