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(τ_1, τ_2)-continuity for multifunctions

Napassanan Srisarakham[1], **Supannee Sompong**[2], **Chawalit Boonpok**[1]

[1]Mathematics and Applied Mathematics Research Unit
Department of Mathematics
Faculty of Science
Mahasarakham University
Maha Sarakham, 44150, Thailand

[2]Department of Mathematics and Statistics
Faculty of Science and Technology
Sakon Nakhon Rajbhat University
Sakon Nakhon, 47000, Thailand

email: napassanan.sri@msu.ac.th, s_sompong@snru.ac.th,
chawalit.b@msu.ac.th

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Abstract

In this paper, we introduce the notion of (τ_1, τ_2)-continuous multifunctions. Moreover, we investigate some characterizations of (τ_1, τ_2)-continuous multifunctions.

1 Introduction

As a field of mathematics, Topology is concerned with all questions directly or indirectly related to continuity. Continuity of functions is an important notion for the study and investigation in the theory of classical point set topology. This notion has been extended to the setting of multifunctions

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The corresponding author is Napassanan Srisarakham.
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and has been generalized by weaker forms of open sets. Noiri and Popa [7] investigated the notions upper and lower M-continuous multifunctions as multifunctions defined between sets satisfying certain minimal conditions. Popa and Noiri [8] introduced and studied the notion of m-continuous multifunctions. Laprom et al. [6] introduced and investigated the notion of $\beta(\tau_1, \tau_2)$-continuous multifunctions. Moreover, several characterizations of $(\tau_1, \tau_2)\alpha$-continuous multifunctions, $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions and almost weakly (τ_1, τ_2)-continuous multifunctions were established in [9], [4] and [3], respectively. In this paper, we introduce the notion of (τ_1, τ_2)-continuous multifunctions. We also investigate some characterizations of (τ_1, τ_2)-continuous multifunctions.

2 Preliminaries

Throughout the paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [5] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [5] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [5] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [5] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3 Characterizations of (τ_1, τ_2)-continuous multifunctions

We begin this section by introducing the notion of (τ_1, τ_2)-continuous multifunctions.

Definition 3.1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open sets V_1, V_2 of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$.*

Theorem 3.2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is (τ_1, τ_2)-continuous;

(2) $x \in \tau_1\tau_2\text{-}Int(F^+(V_1) \cap F^-(V_2))$ for every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$;

(3) $F^+(V_1) \cap F^-(V_2)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y;

(4) $F^-(K_1) \cup F^+(K_2)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-closed sets K_1, K_2 of Y;

(5) $\tau_1\tau_2\text{-}Cl(F^-(B_1) \cup F^+(B_2)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B_1)) \cup F^+(\sigma_1\sigma_2\text{-}Cl(B_2))$ for every subsets B_1, B_2 of Y;

(6) $F^-(\sigma_1\sigma_2\text{-}Int(B_1)) \cap F^+(\sigma_1\sigma_2\text{-}Int(B_2)) \subseteq \tau_1\tau_2\text{-}Int(F^-(B_1) \cap F^+(B_2))$ for every subsets B_1, B_2 of Y.

Proof. (1) $\Rightarrow$ (2): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. Then, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$. Thus, $U \subseteq F^+(V_1) \cap F^-(V_2)$ and hence $x \in \tau_1\tau_2\text{-Int}(F^+(V_1) \cap F^-(V_2))$.

(2) ⇒ (3): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y and $x \in F^+(V_1) \cap F^-(V_2)$. Then $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. By (2), $x \in \tau_1\tau_2\text{-Int}(F^+(V_1) \cap F^-(V_2))$ and hence $F^+(V_1) \cap F^-(V_2) \subseteq \tau_1\tau_2\text{-Int}(F^+(V_1) \cap F^-(V_2))$. This shows that $F^+(V_1) \cap F^-(V_2)$ is $\tau_1\tau_2$-open in X.

(3) ⇒ (4): This follows from the fact that $F^-(Y - B) = X - F^+(B)$ and $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(4) ⇒ (5): Let B_1, B_2 be any subsets of Y. Then $\sigma_1\sigma_2\text{-Cl}(B_1)$ and $\sigma_1\sigma_2\text{-Cl}(B_2)$ are $\sigma_1\sigma_2$-closed in Y. By (4), we have

$$\begin{aligned}\tau_1\tau_2\text{-Cl}(F^-(B_1) \cup F^+(B_2)) &\subseteq \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(B_1)) \cup F^+(\sigma_1\sigma_2\text{-Cl}(B_2))) \\ &= F^-(\sigma_1\sigma_2\text{-Cl}(B_1)) \cup F^+(\sigma_1\sigma_2\text{-Cl}(B_2)).\end{aligned}$$

(5) ⇒ (6): Let B_1, B_2 be any subsets of Y. By (5), we have

$$\begin{aligned}&X - \tau_1\tau_2\text{-Int}(F^-(B_1) \cap F^+(B_2)) \\ &= \tau_1\tau_2\text{-Cl}(X - (F^-(B_1) \cap F^+(B_2))) \\ &= \tau_1\tau_2\text{-Cl}(F^+(Y - B_1) \cup F^-(Y - B_2)) \\ &\subseteq F^+(\sigma_1\sigma_2\text{-Cl}(Y - B_1)) \cup F^-(\sigma_1\sigma_2\text{-Cl}(Y - B_2)) \\ &= F^+(Y - \sigma_1\sigma_2\text{-Int}(B_1)) \cup F^-(Y - \sigma_1\sigma_2\text{-Int}(B_2)) \\ &= X - (F^-(\sigma_1\sigma_2\text{-Int}(B_1)) \cap F^+(\sigma_1\sigma_2\text{-Int}(B_2)))\end{aligned}$$

and hence $F^-(\sigma_1\sigma_2\text{-Int}(B_1)) \cap F^+(\sigma_1\sigma_2\text{-Int}(B_2)) \subseteq \tau_1\tau_2\text{-Int}(F^-(B_1) \cap F^+(B_2))$.

(6) ⇒ (1): Let $x \in X$ and V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. By (6), we have

$$F^+(V_1) \cap F^-(V_2) \subseteq \tau_1\tau_2\text{-Int}(F^+(V_1) \cap F^-(V_2)).$$

Now, put $U = F^+(V_1) \cap F^-(V_2)$, then U is a $\tau_1\tau_2$-open set of X containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for each $z \in U$. This shows that F is (τ_1, τ_2)-continuous. □

Definition 3.3. [2] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous at a point $x \in X$ if for and each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous if f has this property at each point of X.*

Corollary 3.4. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is (τ_1, τ_2)-continuous;

(2) $x \in \tau_1\tau_2\text{-}Int(f^{-1}(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y *containing* $f(x)$*;*

(3) $f^{-1}(V)$ *is* $\tau_1\tau_2$*-open in* X *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(4) $f^{-1}(K)$ *is* $\tau_1\tau_2$*-closed in* X *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(5) $\tau_1\tau_2\text{-}Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(6) $f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(B))$ *for every subset* B *of* Y*.*

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References

[1] C. Berge, Espaces topologiques fonctions multivoques, Dunod, Paris, 1959.

[2] C. Boonpok, N. Srisarakham, (τ_1, τ_2)-continuity for functions, Asia Pac. J. Math., **11,** (2024), 21.

[3] C. Boonpok, C. Viriyapong, Upper and lower almost weak (τ_1, τ_2)-continuity, Eur. J. Pure Appl. Math., **14,** no. 4, (2021), 1212–1225.

[4] C. Boonpok, $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, Heliyon, **6,** (2020), e05367.

[5] C. Boonpok, C. Viriyapong, M. Thongmoon, On upper and lower (τ_1, τ_2)-precontinuous multifunctions, J. Math. Comput. Sci., **18,** (2018), 282–293.

[6] K. Laprom, C. Boonpok, C. Viriyapong, $\beta(\tau_1, \tau_2)$-continuous multifunctions on bitopological spaces, J. Math., **2020** (2020), 4020971.

[7] T. Noiri and V. Popa, On upper and lower M-continuous multifunctions, Filomat, **14,** (2000), 73–86.

[8] V. Popa, T. Noiri, On m-continuous multifunctions, Bul. St. Univ. "Politehnica", Ser. Mat. Fiz. Timişoara, **46 (60)** (2001), 1–12.

[9] C. Viriyapong, C. Boonpok, $(\tau_1, \tau_2)\alpha$-continuity for multifunctions, J. Math., **2020** (2020), 6285763.