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On Quasi-ideals and Bi-ideals of Hypersemigroups and Ternary Hypersemigroups

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Abstract

It is known that if S is a regular semigroup or a regular ordered semigroup, then the quasi-ideals and the bi-ideals of S coincide. The converse does not hold in general [4]. The purpose of this paper is to show that the results mentioned are true for hypersemigroups and ternary hypersemigroups.

1 Introduction

A non-empty set S together with a binary operation $\cdot$ on S, denoted by xy for any x, y in S, is called a *semigroup* if $x(yz) = (xy)z$ for any x, y, z in S. For non-empty subsets A and B of a semigroup S, the product AB is defined to be a subset of S such that $AB = \{ab \mid a \in A, b \in B\}$. A non-empty subset Q of a semigroup S is called a *quasi-ideal* (of S) if $QS \cap SQ \subseteq Q$, and a non-empty subset B of S is called a *bi-ideal* (of S) if $BB \subseteq B$ and if $BSB \subseteq B$. Observe that every quasi-ideal of S is a bi-ideal of S, and a bi-ideal of S need not be a quasi-ideal of S. An element a of a semigroup S is said to

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be *regular* if $a = axa$ for some x in S, and S is said to be *regular* if every element of S is regular. It is known that if S is a regular semigroup, then the quasi-ideals and the bi-ideals of S coincide, and the converse statement does not hold in general. It is also known that the results on any semigroup mentioned before are also true for any ordered semigroup. An *ordered semigroup* $(S, \cdot, \leq)$ consists of a semigroup $(S, \cdot)$ together with a partial order $\leq$ on S that is compatible with the semigroup operation. That is, for any x, y, z in S, if $x \leq y$, then $zx \leq zy$ and $xz \leq yz$. A non-empty subset Q of an ordered semigroup S is called a *quasi-ideal* of S if $(QS) \cap (SQ) \subseteq Q$ and for any x in Q and y in S, if $y \leq x$, then $y \in Q$. A non-empty subset B of S is called a *bi-ideal* of S if $BB \subseteq B$, $BSB \subseteq B$, and for any x in B and y in S, if $y \leq x$, then $y \in B$. Observe that every quasi-ideal of S is a bi-ideal of S, and a bi-ideal of S need not be a quasi-ideal of S. An element a of an ordered semigroup S is said to be *regular* if $a \leq axa$ for some x in S, and S is said to be *regular* if every element of S is regular. It is known that if S is a regular ordered semigroup, then the quasi-ideals and the bi-ideals of S coincide, and the converse statement does not hold in general [4].

Algebraic hyperstructures have been studied extensively and several applications have also been investigated [1], [2].

The purpose of this paper is to show that the known results mentioned above are also true for hypersemigroups and for ternary hypersemigroups.

2 On hypersemigroups

Let S be a non-empty set and let $P(S)$ denote the set of all subsets of S. A mapping $\circ : S \times S \to P(S) \setminus \{\emptyset\}$ is called a *hyperoperation* on S, and a pair $(S, \circ)$ is called a *hypergroupoid*. For non-empty subsets A and B of S, let

$$A \circ B = \cup\{a \circ b \mid a \in A, b \in B\}.$$

If $x \in S$, we write $A \circ x$ for $A \circ \{x\}$ and write $x \circ A$ for $\{x\} \circ A$.

A hypergroupoid $(S, \circ)$ is said to be a *hypersemigroup* if

$$x \circ (y \circ z) = (x \circ y) \circ z$$

for any x, y, z in S. That is,

$$\bigcup_{w \in y \circ z} x \circ w = \bigcup_{v \in x \circ y} v \circ z$$

for any x, y, z in S.

A non-empty subset Q of a hypersemigroup S is called a *quasi-ideal* (of S) if $(Q \circ S) \cap (S \circ Q) \subseteq Q$, and a non-empty subset B of S is called a *bi-ideal* of S if $B \circ B \subseteq B$ and if $B \circ S \circ B \subseteq B$. Observe that every quasi-ideal of S is a bi-ideal of S. Indeed, if Q is a quasi-ideal of S, then

$$Q \circ Q \subseteq (Q \circ S) \cap (S \circ Q) \subseteq Q \text{ and } QSQ \subseteq (Q \circ S) \cap (S \circ Q) \subseteq Q.$$

Hence Q is a bi-ideal of S. An element a of a hypersemigroup S is said to be *regular* if $a \in a \circ x \circ a$ for some x in S, and S is said to be a *regular hypersemigroup* if every element of S is regular.

Theorem 2.1. *If $(S, \circ)$ is a regular hypersemigroup, then the quasi-ideals and the bi-ideals of S coincide.*

Proof. Assume $(S, \circ)$ is a regular hypersemigroup. Let B be a bi-ideal of S. To show that $(B \circ S) \cap (S \circ B) \subseteq B$, let $x \in (B \circ S) \cap (S \circ B)$. By assumption, there exists $y \in S$ such that $x \in x \circ y \circ x$. This implies

$$x \in (B \circ S) \circ S \circ (S \circ B) \subseteq B \circ S \circ B \subseteq B.$$

Therefore, B is a quasi-ideal of S. Hence the quasi-ideals and the bi-ideals of S coincide. □

Remark 2.2. *The converse of Theorem 2.1 does not hold in general as the following example shows.*

Example: Let $S = \{a, b, c, d, e\}$ be a hypersemigroup with a hyperoperation defined by:

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{b, e\}$	$\{c, a\}$	$\{b, e\}$	$\{b, e\}$
b	$\{b, e\}$	$\{b, e\}$	$\{b, e\}$	$\{b, e\}$	$\{b, e\}$
c	$\{a\}$	$\{b, e\}$	$\{c, a\}$	$\{b, e\}$	$\{b, e\}$
d	$\{d, b, e\}$	$\{b, e\}$	$\{d, b, e\}$	$\{b, e\}$	$\{b, e\}$
e	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$

Since d is not regular, we have that S is not regular. The quasi-ideals of S are $\{e\}, \{b, e\}, \{a, b, e\}, \{a, b, c, e\}, \{b, d, e\}, \{a, b, d, e\}, S$. The bi-ideals and the quasi-ideals of S coincide.

3 On ternary hypersemigroups

Let T be a non-empty set and let $P(T)$ denote the set of all subsets of H. A map $f : T \times T \times T \to P(T) \setminus \{\emptyset\}$ is called a *ternary hyperoperation* on T, and (T, f) is called a *ternary hypergroupoid*. For non-empty subsets A, B, and C of T, the set product $f(A, B, C)$ is defined by:

$$f(A, B, C) = \cup\{f(a, b, c) \in P(T) \setminus \{\emptyset\} \mid a \in A, b \in B, c \in C\}.$$

For example, we write $f(A, B, x)$ for $f(A, B, \{x\})$, $f(A, y, x)$ for $f(A, \{y\}, \{x\})$, and $f(x, B, C)$ for $f(\{x\}, B, C)$.

A ternary hypergroupoid (T, f) is called a *ternary hypersemigroup* if

$$f(f(x_1, x_2, x_3), x_4, x_5) = f(x_1, f(x_2, x_3, x_4), x_5) = f(x_1, x_2, f(x_3, x_4, x_5))$$

for any x_1, x_2, x_3, x_4, x_5 in T [5]. A non-empty subset H of a ternary hypersemigroup T is called a *ternary subhypersemigroup* of T if $f(H, H, H) \subseteq H$. A non-empty subset I of a ternary hypersemigroup T is called a *left* (resp. right, lateral) *ideal* (of T) if $f(T, T, I) \subseteq I$ (resp. $f(I, T, T) \subseteq I$, $f(T, I, T) \subseteq I$). Observe that $f(a, T, T) \cup a$ (resp. $f(T, a, T) \cup f(T, T, a, T, T) \cup a$, $f(T, T, a) \cup a$) is a left (resp. right, lateral) ideal of T. For non-empty subsets X_1, X_2, X_3, X_4, X_5 of T, we have that

$$f(f(X_1, X_2, X_3), X_4, X_5) = f(X_1, f(X_2, X_3, X_4), X_5) = f(X_1, X_2, f(X_3, X_4, X_5));$$

then we write

$$f(X_1, X_2, X_3, X_4, X_5)$$

for $f(f(X_1, X_2, X_3), X_4, X_5)$. A non-empty subset Q of a ternary hypersemigroup T is called a *quasi-ideal* (of T) if

$$f(Q, T, T) \cap (f(T, Q, T) \cup f(T, T, Q, T, T)) \cap f(T, T, Q) \subseteq Q.$$

This is equivalent to

$$f(Q, T, T) \cap f(T, Q, T) \cap f(T, T, Q) \subseteq Q$$

and

$$f(Q, T, T) \cap f(T, T, Q, T, T) \cap f(T, T, Q) \subseteq Q.$$

A non-empty subset B of a ternary hypersemigroup T is called a *bi-ideal* (of T) if $f(B, B, B) \subseteq B$ and if $f(B, T, B, T, B) \subseteq B$. Observe that every quasi-ideal is a bi-ideal. Indeed, if Q is a quasi-ideal, then

$$f(Q,Q,Q) \subseteq f(Q,T,T) \cap f(T,Q,T) \cap f(T,T,Q) \subseteq Q \text{ and}$$
$$f(Q,T,Q,T,Q) \subseteq f(Q,T,T) \cap f(T,T,Q,T,T) \cap f(T,T,Q) \subseteq Q.$$

Hence Q is a bi-ideal. An element a of a ternary hypersemigroup T is said to be *regular* if $a \in f(a,x,a,y,a)$ for some x, y in T, and T is said to be a *regular ternary hypersemigroup* if every element of T is regular.

Theorem 3.1. *A ternary hypersemigroup (T,f) is regular if and only if $f(R,M,L) = R \cap M \cap L$ for any right ideal R, lateral ideal M, and left ideal L of T.*

Proof. Assume that $f(R,M,L) = R \cap M \cap L$ for any right ideal R, lateral ideal M, and left ideal L of T. Let $a \in T$. Setting $R = f(a,T,T) \cup a, M = f(T,a,T) \cup f(T,T,a,T,T) \cup a, L = f(T,T,a) \cup a$. Then $a \in R \cap M \cap L = f(R,M,L)$. We have $a \in f(f(a,T,T) \cup a, f(T,a,T) \cup f(T,T,a,T,T) \cup a, f(T,T,a) \cup a) \subseteq f(a,T,a)$. So $a \in f(a,T,a)$, and hence there exists an element $x \in T$ such that $a \in f(a,x,a)$. This implies that a is regular and hence T is regular.

Assume T is a regular ternary hypersemigroup. Let R, M and L be a right ideal, a lateral ideal and a left ideal, respectively. Then $f(R,M,L) \subseteq R \cap M \cap L$. Let $a \in R \cap M \cap L$. By assumption, we have $a \in f(a,x,a)$ for some $x \in T$. We have $a \in f(a,x,a) = f(f(a,x,a), f(x,a,x), f(a,x,a)) \in f(R,M,L)$. Thus $R \cap M \cap L \subseteq f(R,M,L)$. So $f(R,M,L) = R \cap M \cap L$. ☐

Theorem 3.2. *Let (T,f) be a regular ternary hypersemigroup and Q a non-empty subset of T. Then Q is a quasi-ideal if and only if $f(Q,T,Q,T,Q) \cap f(Q,T,T,Q,T,T,Q) \subseteq Q$.*

Proof. Assume $f(Q,T,Q,T,Q) \cap f(Q,T,T,Q,T,T,Q) \subseteq Q$. Consider

$$\begin{aligned}
& f(Q,T,T) \cap (f(T,Q,T) \cup f(T,T,Q,T,T)) \cap f(T,T,Q) \\
= \ & f(f(Q,T,T), (f(T,Q,T) \cup f(T,T,Q,T,T)), f(T,T,Q) \\
= \ & f(Q,T,T,T,Q,T,T,T,Q) \cup f(Q,T,T,T,T,Q,T,T,T,T,Q) \\
\subseteq \ & f(Q,T,Q,T,Q) \cup f(Q,T,T,Q,T,T,Q) \\
\subseteq \ & Q.
\end{aligned}$$

Therefore, Q is a quasi-ideal. Conversely, if Q is a quasi-ideal, then

$f(Q,T,T) \cap f(T,Q,T) \cap f(T,T,Q) \subseteq Q$. Consider

$$\begin{aligned} & f(Q,T,Q,T,Q) \cap f(Q,T,T,Q,T,T,Q) \\ \subseteq\ & f(Q,T,T,Q,T,T,Q) \\ \subseteq\ & f(Q,T,T) \cap f(T,Q,T) \cap f(T,T,Q) \\ \subseteq\ & Q. \end{aligned}$$

We have $f(Q,T,Q,T,Q) \cap f(Q,T,T,Q,T,T,Q) \subseteq Q$. □

Theorem 3.3. *If (T,f) is a regular ternary hypersemigroup, then the quasi-ideals and the bi-ideals of T coincide.*

Proof. If B is a bi-ideal, then $f(B,T,B,T,B) \cap f(B,T,T,B,T,T,B) \subseteq B$. Consequently, by Theorem 3.2, B is a quasi-ideal. □

Remark 3.4. *The converse of Theorem 3.3 does not hold in general as the following example shows.*

Example: Let $T' = \{a,b,c,d,e\}$ be a ternary hypersemigroup with a ternary hyperoperation f defined by:

f	a	b	c	d	e
aa	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
ab	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
ac	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
ad	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
ae	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$

f	a	b	c	d	e
ba	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
bb	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
bc	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
bd	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
be	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$

f	a	b	c	d	e
ca	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
cb	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
cc	$\{a\}$	$\{a\}$	$\{a,b,c\}$	$\{a\}$	$\{a,b,c\}$
cd	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
ce	$\{a\}$	$\{a\}$	$\{a,b,c\}$	$\{a\}$	$\{a,b,c\}$

f	a	b	c	d	e
da	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$
db	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$
dc	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$
dd	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$
de	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$

f	a	b	c	d	e
ea	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$
eb	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$
ec	$\{a,b,d\}$	$\{a,b,d\}$	T'	$\{a,b,d\}$	T'
ed	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$
ee	$\{a,b,d\}$	$\{a,b,d\}$	T'	$\{a,b,d\}$	T'

Since b is not regular, we have that T' is not regular. The quasi-ideals of T' are $\{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, \{a,b,c,d\}, T'$. The bi-ideals and the quasi-ideals of T' coincide.

Remark 3.5. *The results in this section hold true for any ternary semigroup [3].*

Example: Let $T'' = \{a,b,c,d,e\}$ be a ternary semigroup with a ternary operation f defined by:

f	a	b	c	d	e
aa	a	b	c	a	e
ab	b	c	b	b	b
ac	c	b	c	c	c
ad	a	b	c	a	e
ae	c	b	c	c	c

f	a	b	c	d	e
ba	b	c	b	b	b
bb	c	b	c	c	c
bc	b	c	b	b	b
bd	b	c	b	b	b
be	b	c	b	b	b

f	a	b	c	d	e
ca	c	b	c	c	c
cb	b	c	b	b	b
cc	c	b	c	c	c
cd	c	b	c	c	c
ce	c	b	c	c	c

f	a	b	c	d	e
da	d	b	c	d	e
db	b	c	b	b	b
dc	c	b	c	c	c
dd	d	b	c	d	e
de	c	b	c	c	c

f	a	b	c	d	e
ea	c	b	c	c	c
eb	b	c	b	b	b
ec	c	b	c	c	c
ed	c	b	c	c	c
ee	c	b	c	c	c

Since e is not regular, we have that T'' is not regular. The quasi-ideals of T'' are $\{b,c\}, \{a,b,c\}, \{b,c,d\}, \{a,b,c,d\}, \{b,c,e\}, \{a,b,c,e\}, \{b,c,d,e\}, T''$. The bi-ideals and the quasi-ideals of T'' coincide.

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