International Journal of Mathematics and
Computer Science, **19**(2024), no. 4, 1101–1104

On the Exponential Diophantine Equation $8^x + 161^y = z^2$

K. Manikandan, R. Venkatraman

Department of Mathematics
Faculty of Engineering and Technology
SRM Institute of Science and Technology, Vadapalani Campus
No. 1, Jawaharlal Nehru Salai, Vadapalani
Chennai-600026, Tamilnadu, India.

email: mk8408@srmist.edu.in, manishkm4439@gmail.com,
venkatrr1@srmist.edu.in, venkarmeeraj@gmail.com

(Received March 4, 2024, Accepted April 4, 2024,
Published June 1, 2024)

Abstract

In this paper, we present a comprehensive analysis of the Diophantine equation $8^x + 161^y = z^2$, focusing on solutions within the domain of non-negative integers. We demonstrate that this equation possesses precisely two solutions; namely, $(1, 0, 3)$ and $(1, 1, 13)$.

1 Introduction

In recent years, mathematicians have delved into Exponential Diophantine equations, like $a^x + b^y = z^2$, where $(a, b, x, y, z) \in \mathbb{Z}^+$. In 2017, Asthana [4] explored the equation $8^x + 113^y = z^2$, revealing that $(1, 0, 3)$, $(1, 1, 11)$, $(3, 1, 25)$ are the only non-negative integer solutions. In 2020, Orosram and Comemuang [5] solved $8^x+n^y = z^2$, obtaining the unique non-negative integer solution $(1, 0, 3)$. In 2023, Manikandan and Venkatraman [2] use various methods to obtain integral solutions to the equation by thoroughly analyzing the properties and characteristics of these solutions. In 2024, Malavika N.

Key words and phrases: Exponential Diophantine equation.
AMS (MOS) Subject Classifications: 11D61.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

and Venkatraman [3] explored the Diophantine equation $3^x + 121^y = z^2$, discovering the only two solutions $(1, 0, 2), (5, 2, 122)$ for non-negative integers x, y, z. Approaches relied on number theory principles, including Mihăilescu's [1] solution in 2004 of Catalan's conjecture.

2 Prerequisites

In this section, we recall Catalan's Conjecture from 1844, which was subsequently proved by Mihăilescu in 2004.

Theorem 2.1 (Mihăilescu's Theorem). Catalan's conjecture is correct. This means that the Diophantine equation $a^x - b^y = 1$ has only one solution, which is $(a, b, x, y) = (3, 2, 2, 3)$. Here, a, b, x, and y are integers where the smallest among them is greater than 1.

Lemma 2.2. [4] The exponential Diophantine equation $8^x + 1 = z^2$ possesses only one solution $(1, 3)$ within the set of non-negative integers x and z.

Lemma 2.3. The exponential Diophantine equation $1 + 161^y = z^2$ does not have any solutions within the set of non-negative integers y and z.

Proof. $y = 0$ leads to the contradiction $z^2 = 2$. Now, we consider the remaining case where $y \geq 1$. According to the proven Catalan's Conjecture, we must have $y = 1$ and so $z^2 = 162$, which is impossible. This concludes the proof. □

3 Main results

Theorem 3.1: The exponential Diophantine equation $8^x + 161^y = z^2$ has exactly two non-negative integer solutions; namely, $(1, 0, 3)$ and $(1, 1, 13)$.

Proof. Let x, y, and z be non-negative integers satisfying the equation $8^x + 161^y = z^2$. By Lemma 2.3, $x \geq 1$. We consider three cases for y:

Case I: If y is zero, then, by Lemma 2.2, the only solution is $(x, y, z) = (1, 0, 3)$.

Case II: If y is even; say $y = 2r$ for some positive integer r, we explore the equation $2 \cdot 161^k = 2^s(2^{3x-2s} - 1)$, where $z - 161^r = 2^s$ and $z + 161^r = 2^{3x-s}$. Two subcases arise:

1. If $s = 0$, then $z - 161^r = 1$, leading to a contradiction as z is even. 2. If $s = 1$, then $2^{3x-2} - 1 = 161^k$. If $x = 1$, then $161^r = 1$ implying that $r = 0$ and $y = 0$. For $x \geq 2$, Proposition 2.1 implies that $r = 1$, but $2^{3x-2} = 162$, which is impossible.

Case III: If y is odd; say $y = 2r + 1$, we express the equation as $8^x + 17 \cdot 161^{2r} = z^2 - 144 \cdot 161^{2r}$. Factoring, we get $8^x + 17 \cdot 161^{2r} = (z - 12 \cdot 161^r)(z + 12 \cdot 161^r)$. Two possibilities arise:

1. Solving the set of equalities, we find that $(x, y, z) = (1, 1, 13)$. 2. The second set of equalities results in an unsolvable equation.

The proof of our main theorem is now complete. $\square$

Corollary 3.1. *There are no non-negative integer solutions to the Diophantine equation* $8^x + 161^y = v^4$.

Proof. Assuming x, y, and v satisfy $8^x + 161^y = v^4$, let $z = v^2$. Theorem 3.1 gives the possible solutions $(x, y, z) \in \{(1, 0, 3), (1, 1, 13)\}$. However, $v^2 = z$ cannot be 3 or 13 as they are not perfect squares. Consequently, the equation has no solution in non-negative integers. $\square$

Corollary 3.2. $(1, 0, 1)$ *is the unique solution to the Diophantine equation* $8^x + 161^y = 9v^4$, *where* x, y, *and* v *are non-negative integers.*

Proof. For non-negative integers x, y, and v satisfying $8^x + 161^y = 9v^4$, setting $z = 3v^2$ gives $8^x + 161^y = z^2$. Using Theorem 3.1, the solution is $(x, y, z) = (1, 0, 3)$. As a result, $v = 1$ yielding the unique solution $(1, 0, 1)$. $\square$

4 Conclusion

This study establishes that the Exponential Diophantine equation $8^x + 161^y = z^2$ yields precisely the two solutions $(1, 0, 3)$ and $(1, 1, 13)$ among the non-negative integers.

References

[1] P. Mihăilescu, Primary cyclotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.*, 572, (2004), 167–195.

[2] K. Manikandan, R. Venkatraman, Integral Solutions of The Ternary Diophantine Equation $945(x^2+y^2)-1889xy+10(x+y)+100 = pz^q$, *IAENG International Journal of Applied Mathematics* , **53**, (2023), 1657–1672.

[3] Malavika N., R. Venkatraman, On the Exponential Diophantine Equation $3^x + 121^y = z^2$, *Int. J. Math. Comput. Sci.*, **19,** no. 3, (2024), 917–920.

[4] Shivangi Asthana, On the Diophantine Equation $8^x + 113^y = z^2$, *Int. J. Algebra.*, **11,** no. 5, (2017), 225–230.

[5] Wachirarak Orosram, Chalermwut Comemuang, On the Diophantine equation $8^x + n^y = z^2$, *WSEAS Transactions on Mathematics*, **19,** (2020), 520–522.