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An Exponential Diophantine Equation $x^2 + 3^a 97^b = y^n$

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Abstract

In this paper, we determine all positive integer solutions (x, y, n, a, b) of the equation $x^2 + 3^a 97^b = y^n$ for non-negative integers a and b under the condition that the non-negative integers x and y are relatively prime, $n \geq 3$.

1 Introduction

In recent years, many papers dealt with the Diophantine equation

$$x^2 + p_1^{\alpha_1} \dots p_k^{\alpha_k} = y^n, \quad n \geq 3, \quad \gcd(x, y) = 1$$

in non-negative integers $(x, y, \alpha_1, \dots, \alpha_k)$, where $p_i's$ are fixed prime numbers. With the development of modern tools such as the Primitive Divisor Theorem, the modular approach, and the computational techniques, many authors investigated the above equation when $k \geq 1$. In particular, the cases $(p_1, p_2) \in \{(2, 3), (3, 41)\}$ were considered in [3, 1].

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2 Preliminaries

The Lucas pair, denoted as $(\eta, \overline{\eta})$, involves algebraic integers satisfying specific conditions. These pairs generate Lucas numbers through the formula $L_n(\eta, \overline{\eta}) = \frac{\eta^n - \overline{\eta}^n}{\eta - \overline{\eta}}$. The existence of primitive divisors for these Lucas numbers is vital. A prime number p is considered a primitive divisor of $L_n(\eta, \overline{\eta})$ if $p \mid L_n(\eta, \overline{\eta})$ and $p \nmid (\eta - \overline{\eta})^2 \prod_{i=1}^{n-1} L_i(\eta, \overline{\eta})$ for $n > 1$. In addition, a primitive divisor q satisfies $q \equiv \left(\frac{(\eta-\overline{\eta})^2}{q}\right) \pmod{n}$, where $(\frac{*}{q})$ denotes the Legendre symbol. For $n > 4$ and $n \neq 6$, all n-th terms of Lucas sequences have primitive divisors, excluding specific values of $\eta, \overline{\eta}$, and n [2].

3 Main Result

Theorem 3.1. *For $n \geq 3$, $a, b \geq 0$, the only solutions of the equation*

$$x^2 + 3^a 97^b = y^n, \ x, y \geq 1, \ \gcd(x, y) = 1 \tag{3.1}$$

are given by $(x, y, a, b) \in \{(1405096, 12545, 0, 2), (530, 79, 7, 1), (1144, 115, 7, 1), (48664, 1555, 15, 1), (46, 13, 4, 0), (730, 109, 4, 2), (10, 7, 5, 0), (3428, 307, 11, 1), (1034, 103, 5, 1), (7730, 391, 5, 1), (871680070, 912511, 11, 3)\}$ *when* $n = 3$ *and* $(x, y, a, b) = (48, 7, 0, 1)$ *when* $n = 4$.

Proof. The proof of the theorem will be investigated for cases of $n = 3$, $n = 4$, and $n \geq 5$ as follows:
Case 1: When $n = 3$, let $a = 6a_1 + i$ and $b = 6b_1 + j$, where $i, j \in \{0, 1, \ldots, 5\}$. Equation (3.1) transforms into the elliptic curve $L^2 = M^3 - 3^i 97^j$ with $L = \frac{x}{3^{3a_1} 97^{3b_1}}$ and $M = \frac{y}{3^{2a_1} 97^{2b_1}}$. Therefore, we need to calculate all $\{3, 97\}$-integral points on 36 elliptic curves for each i and j. We use MAGMA to determine all the $\{3, 97\}$- integral points on the above elliptic curves satisfying $gcd(numerator(L), numerator(M)) = 1$ and check for the desired solutions. All the solutions are $(x, y, a, b) \in \{(1405096, 12545, 0, 2), (530, 79, 7, 1), (1144, 115, 7, 1), (48664, 1555, 15, 1), (46, 13, 4, 0), (730, 109, 4, 2), (10, 7, 5, 0), (3428, 307, 11, 1), (1034, 103, 5, 1), (7730, 391, 5, 1), (871680070, 912511, 11, 3)\}$.
Case 2: When $n = 4$, let $a = 4\alpha_1 + i$ and $b = 4\beta_1 + j$, where $i, j \in \{0, 1, 2, 3\}$. Equation (3.1) transforms into $A^2 = B^4 - 3^i 97^j$, with $A = \frac{x}{3^{2\alpha_1} 97^{2\beta_1}}$ and $B = \frac{y}{3^{\alpha_1} 97^{\beta_1}}$. Identifying $\{3, 97\}$-integral points on 16 quartic curves corresponds to finding integer solutions of Equation (3.1). Using SIntegralLjunggrenPoints, we determine all S-integral points on these curves, resulting in $(A, B, i, j) = (\mp 1, 0, 0, 0), (\mp 7, 48, 0, 1)$. Considering the condition on x and y, Equation (3.1) has a solution $(x, y, a, b) = (48, 7, 0, 1)$.

Case 3: Now, suppose that $n \geq 5$. If there exists a solution for Equation (3.1) with $n = 2^k$ and $k \geq 3$, it can be obtained from solutions with $n = 4$ since $y^{2^k} = \left(y^{2^{k-2}}\right)^4$. Consequently, there are no solutions for (3.1) with $n = 2^k$ and $k \geq 3$. Similarly, (3.1) has no solution for $n = 3^k$ and $k \geq 2$. Hence, without loss of generality, n is an odd prime. Let's initiate the analysis of the factorization of Equation (3.1) in the field $K = Q(\sqrt{-d})$ as follows: $(x + e\sqrt{-d})(x - e\sqrt{-d}) = y^n$ where $e = 3^\alpha 97^\beta$ for some integers $\alpha, \beta \geq 0$ and $d \in \{1, 3, 97, 291\}$. Assuming that y is even leads to a contradiction as x must be odd according to (3.1) and so $1 + 3^\alpha \equiv 0 \pmod 8$. As a result, y is an odd integer and therefore the ideals formed by $(x+e\sqrt{-d})$ and $(x-e\sqrt{-d})$ are relatively prime in K. The class number $h(K)$ takes on one of two values 1 or 4 for the choice of d. Thus, we can deduce that $\gcd(n, h(d)) = 1$. Since $d \not\equiv 3 \pmod 4$ and n is also coprime to the order of the unit group of $\mathcal{O}_K$, we can write

$$x + e\sqrt{-d} = \xi^n = (s + t\sqrt{-d})^n, \quad x - e\sqrt{-d} = \overline{\xi}^n = (s - t\sqrt{-d})^n$$

and $y = s^2 + dt^2$ for some rational integers s and t. By analyzing these equations, we can derive that $e = L_n t$, where $L_n = \frac{\xi^n - \overline{\xi}^n}{\xi - \overline{\xi}}$. Notably, the sequence L_n is a Lucas sequence. The Lucas sequences without primitive divisors are explicitly enumerated in [2], and it is confirmed that L_n doesn't match any of them. Consequently, we delve into the possibility that a primitive divisor may exist for L_n. Assume q is any primitive divisor of L_n. In this case, q is either 3 or 97. Considering that any primitive divisor $q \equiv \pm 1(\mod n)$, we rule out the possibility $q = 3$ given that $n \geq 5$. Thus, we continue with $q = 97$. According to the definition of a primitive divisor, $q \nmid (\xi - \overline{\xi})^2 = -4dt^2$, indicating that $d = 1$. Moreover, since $\left(\frac{-4t^2}{q}\right) = \left(\frac{-1}{97}\right) = 1$, we deduce that $97 \equiv 1 \pmod n$. Since the only possible values for n are 2 and 3, this contradicts the assumption made in case 3. Consequently, no such n exists for $n \geq 5$. This completes the proof. □

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