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# Generalized Inverses of Linear Relations

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**Abstract**

We develop the notion of a generalized inverse of a linear relation and obtain lower bounds for the minimum modulus of such inverses which arise out of certain perturbations. These lower bounds are used to obtain norm bounds for such generalized inverses.

## 1 Introduction

Let $H_1$ and $H_2$ be Hilbert spaces over the same field of scalars and consider the fundamental problem of solving the multivalued operator inclusion $b \in \mathcal{T}x$ where $b \in H_2$ and $\mathcal{T} : H_1 \to H_2$ is a multivalued operator with closed range. In the event that this inclusion has no solution $x$, it is still possible to assign the best possible solution to the problem. In such a case, it is reasonable to consider as a generalized solution of this inclusion any solution $u$ in $H_1$ of the inclusion $Pb \in \mathcal{T}x$ where $P$ is the projection of $H_2$ onto the range of $\mathcal{T}$. Another natural approach to assigning generalized solutions to the inclusion $b \in \mathcal{T}x$ is to find a $u \in H_1$ which comes closest to solving this inclusion in the sense that dist $(b, \mathcal{T}u) \leq$ dist $(b, \mathcal{T}x)$ for any $x$ in the domain of $\mathcal{T}$. We consider such generalized solutions and use them to develop

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the theory of generalized inverses for multivalued operators, including norm bounds for such inverses arising out of small perturbations.

In all that follows,$H$, $H_1$, and $H_2$ will be Hilbert spaces and any linear subset $\mathcal{T}$ of $H_1 \times H_2$ will be called a linear relation (or a multivalued linear operator) from $H_1$ to $H_2$. We say that $\mathcal{T}$ is closed if it is closed as a subspace of $H_1 \times H_2$. The domain $D(\mathcal{T})$, range $R(\mathcal{T})$, null space $N(\mathcal{T})$, and graph $G(\mathcal{T})$ of a linear relation $\mathcal{T}$ are defined by

$$\begin{aligned} D(\mathcal{T}) &:= \{x \in H_1 : (x, y) \in T \text{ for some } y \in H_2\}, \\ R(\mathcal{T}) &:= \{y \in H_2 : (x, y) \in T \text{ for some } x \in H_1\}, \\ N(\mathcal{T}) &:= \{x \in D(\mathcal{T}) : (x, 0) \in T\}, \\ G(\mathcal{T}) &:= \{(x, y) \in H_1 \times H_2 : x \in D(\mathcal{T}) \text{ and } (x, y) \in T\} \end{aligned}$$

If $x \in D(\mathcal{T})$ we define $\mathcal{T}x$ to be the set $\mathcal{T}x := \{y \in R(\mathcal{T}) : (x, y) \in \mathcal{T}\}$ and $G(\mathcal{T}^{-1}) := \{(y, x) : (x, y) \in G(\mathcal{T})\}$. Hence $N(\mathcal{T}) = \mathcal{T}^{-1}(0)$.

Let $LR\,(H_1, H_2)$ and $CLR\,(H_1, H_2)$ denote the collection of all linear relations and that of closed linear relations from $H_1$ to $H_2$ respectively. Then $\mathcal{T} \in LR\,(H_1, H_2)$ if for any $x, z \in D(\mathcal{T})$ and any nonzero $\alpha \in \mathbb{C}$,

$$\mathcal{T}x + \mathcal{T}z = \mathcal{T}(x + z) \quad \text{and} \quad \alpha\mathcal{T}x = \mathcal{T}(\alpha x),$$

where these equalities are understood to be set equalities. It is well known that if $\mathcal{T} \in LR\,(H_1, H_2)$ then $y \in \mathcal{T}(x)$ if and only if $\mathcal{T}(x) = \mathcal{T}(0) + y$ [1]. For $\mathcal{S}, \mathcal{T} \in LR\,(H_1, H_2)$, we define $(\mathcal{T} + \mathcal{S})x := \{y + z : y \in \mathcal{T}x,\ z \in \mathcal{S}x\}$.

By $B_H$ we mean the set $B_H := \{x \in H : \|x\| \le 1\}$. For a closed linear subspace $E$ of $H$, we denote by $Q_E$ the natural quotient map with domain $H$ and null space $E$. For $\mathcal{T} \in LR\,(H_1, H_2)$, we shall denote $Q_{\overline{\mathcal{T}(0)}}$ by $Q_{\mathcal{T}}$. It is well known that for $\mathcal{T} \in LR\,(H_1, H_2)$, the operator $Q_{\mathcal{T}}\mathcal{T}$ is single valued [1]. For $\mathcal{T} \in LR\,(H_1, H_2)$, we set $\|\mathcal{T}x\| = \|Q_{\mathcal{T}}\mathcal{T}x\|$ for $x \in D(\mathcal{T})$ and $\|\mathcal{T}\| = \|Q_{\mathcal{T}}\mathcal{T}\|$. We say that $\mathcal{T}$ is bounded if $\|\mathcal{T}\| < \infty$ and denote the collection of all such $\mathcal{T}$ by $BLR\,(H_1, H_2)$. We note that if $\mathcal{T}(0) \supset \mathcal{S}(0)$ then $\|\mathcal{T}x + \mathcal{S}x\| \ge \|\mathcal{T}x\| - \|\mathcal{S}x\|$ [5].

If $x$ is an element of $H$ and $M$ is a closed linear subspace of $H$, then the distance $\operatorname{dist}(x, M)$ between $x$ and $M$ is defined by $\operatorname{dist}(x, M) = \inf\limits_{y \in M} \|x - y\|$. If $P_M$ is the orthogonal projection of $H$ onto $M$, then $\operatorname{dist}(x, M) = \|x - P_M x\|$.

The quantity $\gamma(\mathcal{T}) = \inf\left\{\|\mathcal{T}z\| : z \in D(\mathcal{T}) \cap N(\mathcal{T})^{\perp},\ \|z\| = 1\right\}$ is called the minimum modulus of the linear relation $\mathcal{T}$. Note that $\gamma(Q\mathcal{T}) = \gamma(\mathcal{T})$ for $\mathcal{T} \in CLR\,(H_1, H_2)$ [1] and that $\gamma(\mathcal{T}) > 0$ if and only if $R(\mathcal{T})$ is closed [5].

Let $H'$ denote the collection of all bounded linear functionals on a Hilbert space $H$. For $M \subset H$, let $M^{\top} := \{f \in H' : f(m) = 0 \text{ for all } m \in M\}$ and let $M^{\perp} := \{x \in H : \langle x, m\rangle = 0 \text{ for all } m \in M\}$. If $\mathcal{T} \in LR(H_1, H_2)$ and we view $H_1$ and $H_2$ as Banach spaces, then we define the Banach space adjoint $\mathcal{T}'$ of $\mathcal{T}$ to be $\mathcal{T}' \in LR(H_2', H_1')$ defined by

$$(f, g) \in G(\mathcal{T}') \text{ if and only if } f(y) = g(x) \text{ for all } (x, y) \in G(\mathcal{T}).$$

The Hilbert space adjoint $\mathcal{T}^*$ of $\mathcal{T}$ is defined by

$$(h, k) \in G(\mathcal{T}^*) \text{ if and only if } \langle y, h\rangle = \langle x, k\rangle \text{ for all } (x, y) \in G(\mathcal{T}).$$

We see that for $\mathcal{T} \in LR(H_1, H_2)$, $\mathcal{T}'(0) = D(\mathcal{T})^{\top}$, $\mathcal{T}^*(0) = D(\mathcal{T})^{\perp}$, and $N(\mathcal{T}^*) = R(\mathcal{T})^{\perp}$.

Let $\widehat{H}_2 := H_2'/\mathcal{T}'(0)$ and $\widetilde{H}_2 := H_2/\mathcal{T}^*(0)$ and consider the cosets $\widehat{f_z} := f_z + \mathcal{T}'(0)$ and $\widetilde{z} := z + \mathcal{T}^*(0)$, where $z \in H_2$ and $f_z \in H_2'$ are related by $f_z(x) = \langle x, z\rangle$ for $x \in H_2$. We observe from

$$\begin{aligned} w \in \widetilde{z} &\Leftrightarrow w - z \in \mathcal{T}^*(0) \Leftrightarrow \langle x, w - z\rangle = 0 \ \forall x \in D(\mathcal{T}) \\ &\Leftrightarrow \langle x, w\rangle = \langle x, z\rangle \ \forall x \in D(\mathcal{T}) \Leftrightarrow f_w(x) = f_z(x) \ \forall x \in D(\mathcal{T}) \\ &\Leftrightarrow (f_w - f_z)(x) = 0 \ \forall x \in D(\mathcal{T}) \Leftrightarrow f_w - f_z \in D(\mathcal{T})^{\top} = \mathcal{T}'(0) \\ &\Leftrightarrow f_w \in \widehat{f_z}. \end{aligned}$$

that $f_w \in \widehat{f_z}$ if and only if $w \in \widetilde{z}$.

**Lemma 1.1.** *Let $\mathcal{T} \in LR(H_1, H_2)$ be closed and bounded. Then the equalities $\gamma(\mathcal{T}') = \gamma(\mathcal{T}^*)$ and $\|\mathcal{T}'\| = \|\mathcal{T}^*\|$ hold.*

*Proof.* Let $\widehat{f_z}$ and $\widetilde{z}$ be the cosets defined above. The equality $\|f_z\| = \|z\|$ and the observation that $f_w \in \widehat{f_z}$ if and only if $w \in \widetilde{z}$ imply that $\left\|\widehat{f_z}\right\| = \|\widetilde{z}\|$. If $w, z \in H_2$ then we have that $f_w(y) = f_z(x)$ for all $(x, y) \in G(\mathcal{T})$ if and only if $\langle y, w\rangle = \langle x, z\rangle$ for all $(x, y) \in G(\mathcal{T})$. It therefore follows that $f_z \in \mathcal{T}' f_w$ if and only if $z \in \mathcal{T}^* w$. This observation together with the equality $\left\|\widehat{f_z}\right\| = \|\widetilde{z}\|$ imply that for any $z \in H_2$,

$$\left\|Q_{\mathcal{T}'}\mathcal{T}' f_z\right\| = \|Q_{\mathcal{T}^*}\mathcal{T}^* z\| \text{ and } \sup_{\substack{f_z \in D(\mathcal{T}') \\ \|f_z\|=1}} \left\|Q_{\mathcal{T}'}\mathcal{T}' f_z\right\| = \sup_{\substack{z \in D(\mathcal{T}^*) \\ \|z\|=1}} \|Q_{\mathcal{T}^*}\mathcal{T}^* z\|. \quad (1.1)$$

The first and second equalities in the lemma then follow from the first and second equalities in (1.1). □

See [1, Theorems II. 3.2, III. 1.4, III. 4.6] for the equalities.

$$\mathcal{T}^*(0) = D(\mathcal{T})^{\perp}, \ D(\mathcal{T}^*) = \mathcal{T}(0)^{\perp}, \ \gamma(\mathcal{T}) = \gamma(\mathcal{T}^*), \ \& \ \|\mathcal{T}\| = \|\mathcal{T}^*\|.$$

**Lemma 1.2.** [2] *Let $H_1$ and $H_2$ be Hilbert spaces and let $T : H_1 \to H_2$ be a bounded linear operator having closed range. Then there exists a constant $m > 0$ such that $\|Tx\| \geq m\|x\|$ for all $x \in N(T)^\perp$, where $N(T)$ denotes the null space of $T$.*

# 2 Main results

## 2.1 The generalized inverse of a bounded linear relation

**Theorem 2.1.** *Let $\mathcal{T} \in BLR\,(H_1, H_2)$ be such that $R(\mathcal{T})$ is closed and let $P$ denote the projection of $H_2$ onto $R(\mathcal{T})$. Let $b \in \mathcal{H}_2$. Then the following conditions on $u \in \mathcal{H}_1$ are equivalent.*

$$\text{(i) } Pb \in \mathcal{T}u, \quad \text{(ii) } \|z - b\| = \text{dist}\,(b, R(\mathcal{T}))\ \forall z \in \mathcal{T}u, \quad \text{(iii) } \mathcal{T}^*\mathcal{T}u \cap \mathcal{T}^*b \neq \emptyset.$$

*Proof.* (i)$\longrightarrow$(ii): Suppose that $Pb \in \mathcal{T}u$. Since $Pb - b \in R(\mathcal{T})^\perp$, we see that for any $y \in R(\mathcal{T})$,

$$\begin{aligned} \|y - b\|^2 &= \|y - Pb\|^2 + \|Pb - b\|^2 \geq \|Pb - b\|^2 \\ &= \|z - b\|^2 \text{ (where } z = Pb \in \mathcal{T}u). \end{aligned}$$

Equality in (ii) then follows from the definition of dist $(b, R(\mathcal{T}))$ and the fact that $z \in \mathcal{T}u \subset R(\mathcal{T})$.
(ii)$\longrightarrow$(iii): Suppose that $\|z - b\| = \text{dist}\,(b, R(\mathcal{T}))$ for some $z \in \mathcal{T}u$. Since $Pb \in R(\mathcal{T})$, it follows that

$$\|z - b\|^2 = \|z - Pb\|^2 + \|Pb - b\|^2 \geq \|z - Pb\|^2 + \|z - b\|^2$$

and that $z = Pb$. Hence $z - b = Pb - b \in R(\mathcal{T})^\perp = N(\mathcal{T}^*)$ so that $0 \in \mathcal{T}^*(z - b) = \mathcal{T}^*z - \mathcal{T}^*b$. This means that $\mathcal{T}^*\mathcal{T}u \cap \mathcal{T}^*b \neq \emptyset$.
(iii)$\longrightarrow$(i) If (iii) holds then $0 \in \mathcal{T}^*\mathcal{T}u - \mathcal{T}^*b$. This implies that there exists an element $u' \in \mathcal{T}u$ such that $0 \in \mathcal{T}^*u' - \mathcal{T}^*b = T^*(u' - b)$. This means that $\mathcal{T}^*(u' - b) = \mathcal{T}^*(0)$ and that $u' - b \in N(\mathcal{T}^*) = R(\mathcal{T})^\perp$ and therefore $0 = P(u' - b) = u' - Pb$. It therefore follows that $Pb \in \mathcal{T}u$. □

**Definition 2.2.** *A vector $u \in D(\mathcal{T})$ which satisfies the equivalent conditions (i)-(iii) of Theorem 2.1 will be called a least squares solution of the inclusion*

$$b \in \mathcal{T}x. \tag{2.2}$$

Note that since $R(\mathcal{T})$ is closed, a least squares solution of inclusion (2.2) exists for each $b \in \mathcal{H}_2$. If $N(\mathcal{T}) \neq \{0\}$, then there are many least squares solutions of the inclusion (2.2) since if $u$ is a least squares solution then so is $u + v$ for any $v \in N(\mathcal{T})$.

**Lemma 2.3.** *The set* $M = \{u : u \text{ is a least squares solution of } b \in \mathcal{T}x\}$ *is closed and convex for every* $\mathcal{T} \in BLR(H_1, H_2)$.

*Proof.* To show that $M$ is convex, let $\alpha \in [0,1]$ and let $u_1, u_2 \in M$. The linearity of $\mathcal{T}$ implies that

$$\begin{aligned} \mathcal{T}(\alpha u_1 + (1-\alpha)u_2) &= \mathcal{T}(\alpha u_1) + \mathcal{T}((1-\alpha)u_2) = \alpha\mathcal{T}u_1 + \mathcal{T}u_2 - \alpha\mathcal{T}u_2 \\ &\ni \alpha Pb + Pb - \alpha Pb = Pb, \end{aligned}$$

showing that $\alpha u_1 + (1-\alpha)u_2 \in M$ and that $M$ is convex. To show that $M$ is closed, let $(u_n)$ be a sequence in $M$ such that $u_n \to u \in D(\mathcal{T})$. Then

$$\begin{aligned} \|Q\mathcal{T}u_n - Q\mathcal{T}u\| &= \|Q\mathcal{T}(u_n - u)\| = \|\mathcal{T}(u_n - u)\| \\ &\leq \|\mathcal{T}\|\|u_n - u\| \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Since $Pb \in \mathcal{T}u_n \ \forall n$, we see that $Q\mathcal{T}u_n = Q(Pb) = \lim_{n\to\infty} Q\mathcal{T}u_n = Q\mathcal{T}u$, that is, $Q\mathcal{T}(u_n - u) = 0$. It follows from here that for each $n \in \mathbb{N}$, $\mathcal{T}u_n - \mathcal{T}u = \mathcal{T}(u_n - u) = \mathcal{T}(0)$. This means that $\mathcal{T}u_n \cap \mathcal{T}u \neq \emptyset$. Hence $\mathcal{T}u_n = \mathcal{T}u$ for each $n \in \mathbb{N}$. The result then follows immediately. □

We would like to invert $\mathcal{T} \in BLR(\mathcal{H}_1, \mathcal{H}_2)$, by associating each $b \in \mathcal{H}_2$ with some uniquely determined least squares solution $u \in \mathcal{H}_1$. By Lemma 2.3, the set $M$ of all least squares solutions of the inclusion (2.2) is closed and convex and therefore has a unique element of minimal norm. We use this vector of minimal norm to define the generalized inverse of $\mathcal{T}$.

**Definition 2.4.** *Let* $\mathcal{T} \in BLR(\mathcal{H}_1, \mathcal{H}_2)$ *be such that* $R(\mathcal{T})$ *is closed. The relation* $\mathcal{T}^\dagger$ *from* $H_2$ *to* $H_1$ *defined by*

$$\mathcal{T}^\dagger b = u + \mathcal{T}^{-1}(0) \tag{2.3}$$

*where* $u$ *is the least squares solution of minimum norm of the inclusion* (2.2) *is called the generalized inverse of* $\mathcal{T}$.

**Theorem 2.5.** *Let* $\mathcal{T} \in BLR(H_1, H_2)$ *be such that* $R(\mathcal{T})$ *is closed. Then*

$$R(\mathcal{T}^\dagger) \subset N(\mathcal{T})^\perp. \tag{2.4}$$

*Proof.* Let $b \in H_2$ and let $z \in \mathcal{T}^\dagger b$. Then $z = u + v$ where $v \in \mathcal{T}^{-1}(0)$ and $u = u_1 + u_2 \in N(\mathcal{T})^\perp \oplus N(\mathcal{T})$ is the least squares solution of the inclusion $b \in \mathcal{T}x$ of minimal norm. From the definition of $\mathcal{T}^\dagger$, we see that

$$Pb \in \mathcal{T}u = \mathcal{T}(u_1 + u_2) = \mathcal{T}u_1 + \mathcal{T}u_2 = \mathcal{T}u_1 + \mathcal{T}(0) = \mathcal{T}u_1.$$

This shows that $u_1$ is a least squares solution of (2.2). If $u_2 \neq 0$, then $\|u_1\|^2 < \|u_1\|^2 + \|u_2\|^2 = \|u\|^2$, contradicting the fact that $u$ is the least squares solution of minimal norm. Hence $u = u_1 \in N(\mathcal{T})^\perp$. □

**Corollary 2.6.** *Let $\mathcal{T} \in BLR(H_1, H_2)$ be such that $R(\mathcal{T})$ is closed. Then $\mathcal{T}^\dagger \in BLR(H_2, H_1)$.*

*Proof.* Let $\mathcal{T}^\dagger b_1 = u_1 + \mathcal{T}^{-1}(0)$ and $\mathcal{T}^\dagger b_2 = u_2 + \mathcal{T}^{-1}(0)$ where $b_1, b_2 \in \mathcal{H}_2$ and $u_1$ and $u_2$ are least squares solutions (of minimal norm) of the inclusions $b_1 \in \mathcal{T}x$ and $b_2 \in \mathcal{T}x$ respectively. It follows from the linearity of $\mathcal{T}^{-1}(0)$ that $u_1 \in \mathcal{T}^\dagger b_1$ and that $u_2 \in \mathcal{T}^\dagger b_2$. Since $u_1$ and $u_2$ are least squares solutions of the inclusions $b_1 \in \mathcal{T}x$ and $b_2 \in \mathcal{T}x$ respectively, it follows that $Pb_1 \in \mathcal{T}u_1$ and that $Pb_2 \in \mathcal{T}u_2$. Hence

$$P(b_1 + b_2) = Pb_1 + Pb_2 \in \mathcal{T}u_1 + \mathcal{T}u_2 = \mathcal{T}(u_1 + u_2). \quad (2.5)$$

Let $\mathcal{T}^\dagger(b_1 + b_2) = w + \mathcal{T}^{-1}(0)$ where $w$ is the least squares solution of the inclusion $b_1 + b_2 \in \mathcal{T}x$ of minimum norm. Then

$$P(b_1 + b_2) \in \mathcal{T}w. \quad (2.6)$$

It follows from (2.5) and (2.6) that $\mathcal{T}([u_1 + u_2] - w) = \mathcal{T}(0)$ and so, $u_1 + u_2 - w \in N(\mathcal{T})$. Since $u_1 + u_2 - w \in R(\mathcal{T}^\dagger) \subset R(\mathcal{T}^*) = N(\mathcal{T})^\perp$, it follows that $u_1 + u_2 - w = 0$, that is, $u_1 + u_2 = w$. This equality together with the linearity of $\mathcal{T}^{-1}(0)$ imply that $\mathcal{T}^\dagger b_1 + \mathcal{T}^\dagger b_2 = \mathcal{T}^\dagger(b_1 + b_2)$.

We now consider the vector $\alpha b$ where $b \in \mathcal{H}_2$ and $\alpha$ is an arbitrary but fixed scalar. Let

$$\mathcal{T}^\dagger(\alpha b) = v + \mathcal{T}^{-1}(0) \quad (2.7)$$

where $v$ is a least squares solution of minimal norm to the inclusion $\alpha b \in \mathcal{T}x$. Then

$$\alpha Pb = P(\alpha b) \in \mathcal{T}v. \quad (2.8)$$

Similarly, let

$$\mathcal{T}^\dagger(b) = s + \mathcal{T}^{-1}(0) \quad (2.9)$$

where $s$ is a least squares solution of minimal norm to the inclusion $b \in \mathcal{T}x$. Then

$$Pb \in \mathcal{T}s. \tag{2.10}$$

The linearity of $\mathcal{T}$ together with (2.10) imply that

$$\alpha Pb \in \mathcal{T}(\alpha s). \tag{2.11}$$

From (2.8) and (2.11), we conclude that $\mathcal{T}(v - \alpha s) = \mathcal{T}(0)$ and that

$$v - \alpha z \in N(\mathcal{T}). \tag{2.12}$$

Since $v - \alpha z \in N(\mathcal{T})^{\perp}$, (2.12) implies that $v - \alpha z = 0$, that is,

$$v = \alpha z. \tag{2.13}$$

Combining (2.7), (2.9), and (2.13) we obtain

$$\mathcal{T}^{\dagger}(\alpha b) = v + \mathcal{T}^{-1}(0) = \alpha v + \mathcal{T}^{-1}(0) = \alpha \mathcal{T}^{\dagger}(b),$$

which completes the prove that $\mathcal{T}^{\dagger}$ is linear.

For boundedness, we first note $Q_{\mathcal{T}}\mathcal{T}$ is a bounded linear operator with closed range since $\mathcal{T}$ is linear and bounded with closed range. Let $\mathcal{T}^{\dagger}b = u + \mathcal{T}^{-1}(0)$ where $b \in H_2$. Since $u \in \mathcal{T}^{\dagger}b \in R(\mathcal{T}^{\dagger})$ and $N(Q_T T) = N(T)$, we see from Lemma 1.2 and Theorem 2.5 that there exists a constant $\alpha > 0$ such that

$$\|Q_{\mathcal{T}}\mathcal{T}u\| \geq \alpha\|u\|. \tag{2.14}$$

Since $\|u\| \geq \|\widetilde{u}\|$ where $\widetilde{u} = u + N(\mathcal{T})$ is the quotient class of $u$ in the complete space $H_1/N(\mathcal{T}) = H_1/\mathcal{T}^{\dagger}(0)$, we see from (2.14) that

$$\|Q_{\mathcal{T}}\mathcal{T}u\| \geq \alpha\|Q_{\mathcal{T}^{\dagger}}u\| = \alpha\|Q_{\mathcal{T}^{\dagger}}\mathcal{T}^{\dagger}b\|,$$

that is, $\|Q_{\mathcal{T}}\mathcal{T}u\| \geq \alpha\|Q_{\mathcal{T}^{\dagger}}\mathcal{T}^{\dagger}b\|$. Since $Pb \in \mathcal{T}u$, it follows that $\|b\| \geq \|Pb\| \geq \|Q_{\mathcal{T}}\mathcal{T}u\| \geq \alpha\|Q_{\mathcal{T}^{\dagger}}\mathcal{T}^{\dagger}b\|$, and hence $\mathcal{T}^{\dagger}$ is bounded. □

**Lemma 2.7.** *Let $\mathcal{T} \in BLR(H_1, H_2)$. If $R(\mathcal{T})$ is closed, then $T^{\dagger} \in BLR(H_2, H_1)$ is a generalized inverse of $\mathcal{T}$ if and only if*

$$\mathcal{T}^{\dagger}z = w + \mathcal{T}^{-1}(0) \qquad \text{for } w \in N(\mathcal{T})^{\perp} \cap D(\mathcal{T}) \text{ and all } z \in \mathcal{T}w \tag{2.15}$$

*and*

$$\mathcal{T}^{\dagger}y = \mathcal{T}^{-1}(0) = N(\mathcal{T}) \qquad \text{for } y \in R(\mathcal{T})^{\perp}. \tag{2.16}$$

*Proof.* Suppose that (2.15) and (2.16) hold and let $b \in \mathcal{H}_2$ with decomposition

$$b = u + v \in R(\mathcal{T}) \oplus R(\mathcal{T})^{\perp}. \tag{2.17}$$

Then

$$\mathcal{T}^{\dagger} b = \mathcal{T}^{\dagger} u + \mathcal{T}^{\dagger} v = \mathcal{T}^{\dagger} u + \mathcal{T}^{-1}(0). \tag{2.18}$$

Since $u \in R(\mathcal{T})$, $u \in \mathcal{T}(u_1 + u_2)$ for some $u_1 \in N(\mathcal{T})^{\perp}$ and $u_2 \in N(\mathcal{T})$, it follows from $\mathcal{T}(u_1 + u_2) = \mathcal{T}u_1 + \mathcal{T}u_2 = \mathcal{T}u_1 + \mathcal{T}(0) = \mathcal{T}u_1$ that

$$u \in \mathcal{T}u_1 \text{ for } \text{ some } u_1 \in N(\mathcal{T})^{\perp}. \tag{2.19}$$

It therefore follows from (2.15), (2.18) and the linearity of $\mathcal{T}^{-1}(0)$ that

$$\mathcal{T}^{\dagger} b = \mathcal{T}^{\dagger} u + \mathcal{T}^{-1}(0) = u_1 + \mathcal{T}^{-1}(0). \tag{2.20}$$

Equality (2.17) and inclusion (2.19) imply that

$$Pb \in \mathcal{T}u_1 \tag{2.21}$$

and that $u_1$ is a least squares solution of the inclusion $b \in \mathcal{T}x$. It remains to show that $u_1$ is the least squares solution of minimal norm. To do this, assume that $z$ is another least squares solution of the inclusion $b \in \mathcal{T}x$. Then $z - u_1 \in N(\mathcal{T})$ and so $u_1 \perp (z - u_1)$. It therefore follows from $z = u_1 + (z - u_1)$ and that $\|u_1\| \leq \|u_1\| + \|z - u_1\| = \|z\|$. This shows that $u_1$ is the least squares solution of the inclusion $b \in \mathcal{T}x$ of minimal norm and that $\mathcal{T}^{\dagger}$ is the generalized inverse of $\mathcal{T}$.

Now, assume that $\mathcal{T}^{\dagger}$ is the generalised inverse of $\mathcal{T}$. If $y \in R(\mathcal{T})^{\perp}$ then $Py = 0 \in \mathcal{T}(0)$ and therefore $\mathcal{T}^{\dagger} y = 0 + \mathcal{T}^{-1}(0) = \mathcal{T}^{-1}(0)$ since 0 is the least squares solution of the inclusion $y \in \mathcal{T}x$ of minimal norm. Hence, (2.16) holds. Now, suppose that $w \in N(\mathcal{T})^{\perp} \cap D(\mathcal{T})$ and let $z \in \mathcal{T}w$. Then

$$\mathcal{T}^{\dagger} z = v + \mathcal{T}^{-1}(0) \tag{2.22}$$

where $v$ is the least squares solution of the inclusion $z \in \mathcal{T}x$ of minimal norm. Since $z \in R(\mathcal{T})$ and $v$ is a least squares solution of the inclusion $z \in \mathcal{T}x$, it follows that $z \in \mathcal{T}v$. Since $z \in \mathcal{T}w$ and $z \in \mathcal{T}v$, we see that

$$w + \mathcal{T}^{-1}(0) = \mathcal{T}^{-1}(z) = v + \mathcal{T}^{-1}(0). \tag{2.23}$$

It follows from (2.22) and (2.23) that $\mathcal{T}^{\dagger} z = w + \mathcal{T}^{-1}(0)$ so that (2.15) holds. □

**Lemma 2.8.** *Let $\mathcal{T} \in CLR(H_1, H_2)$ be such that $0 < \gamma(\mathcal{T}) < \infty$. Then the equality $\left\|\mathcal{T}^\dagger\right\| = \gamma(\mathcal{T})^{-1}$ holds.*

*Proof.* Let $\mathfrak{D}(\mathcal{T}) = N(\mathcal{T})^\perp \cap D(\mathcal{T})$. The definition of $\mathcal{T}^\dagger$ together with Lemma 2.7 imply that

$$\begin{aligned}
\left\|\mathcal{T}^\dagger\right\| &= \sup_{y\in H_2,\, \|y\|=1} \left\|\mathcal{T}^\dagger y\right\| = \sup_{y\in R(\mathcal{T}),\, \|y\|=1} \left\|\mathcal{T}^\dagger y\right\| = \sup_{z\in R(\mathcal{T}),\, \|z\|\neq 0} \frac{\left\|\mathcal{T}^\dagger z\right\|}{\|z\|} \\
&= \sup_{\substack{0\neq w\in\mathfrak{D}(\mathcal{T})\\ z\in\mathcal{T}w}} \left\{\frac{\text{dist }(w, N(\mathcal{T}))}{\|z\|}\right\} = \sup_{\substack{0\neq w\in\mathfrak{D}(\mathcal{T})\\ \widehat{z}\in\mathcal{T}w \text{ fixed}}} \left\{\frac{\|w\|}{\inf_{z_0\in\mathcal{T}(0)} \|\widehat{z}+z_0\|}\right\} \\
&= \sup_{0\neq w\in\mathfrak{D}(\mathcal{T})} \left\{\frac{\|w\|}{\|\mathcal{T}w\|}\right\} = \left(\inf_{0\neq w\in\mathfrak{D}(\mathcal{T})} \left\{\frac{\|\mathcal{T}w\|}{\|w\|}\right\}\right)^{-1} = \left(\inf_{\substack{x\in\mathfrak{D}(\mathcal{T})\\ \|x\|=1}} \|\mathcal{T}x\|\right)^{-1} \\
&= \gamma(\mathcal{T})^{-1}.
\end{aligned}$$

□

## 2.2 Norm bounds for the generalized inverse

In this section we establish norm bounds for the generalized inverse of a perturbed linear relation. For subspaces $M$ and $N$ of a Hilbert space $H$, define $S_M$ by $S_M := \{x \in M : \|x\| = 1\}$ and consider the quantities $\delta(M, N) = \sup_{x\in S_M} \text{dist}(x, N)$, $\widehat{\delta}(M, N) = \sup_{x\in S_M} \text{dist}(x, S_N)$, and

$$\widehat{\delta}_{\mathcal{T}}(M, N) = \sup_{x\in D(\mathcal{T})\cap S_M} \text{dist}\left(x, D(\mathcal{T}) \cap S_N\right).$$

We see from the inequality $\text{dist}\,(z, S_W) \le 2\ \text{dist}\,(z, W)$, which holds for all $z \in S_H$ [4], that the inequalities $\widehat{\delta}_{\mathcal{T}}(M, N) \le 2\ \delta_{\mathcal{T}}(M, N)$ and $\widehat{\delta}(M, N) \le 2\delta(M, N)$ also hold.

**Lemma 2.9.** *Let $H_1, H_2$ be Hilbert spaces and let $\mathcal{S}, \mathcal{T} \in CLR(H_1, H_2)$ be such that $D(\mathcal{S}) \supset D(\mathcal{T})$, $\mathcal{S}(0) \subset \mathcal{T}(0)$, and $0 < \gamma(\mathcal{T}) < \infty$. Consider the perturbation $\breve{\mathcal{T}} := \mathcal{T} + \mathcal{S}$ of $\mathcal{T}$ by $\mathcal{S}$. Then*

$$\gamma(\breve{\mathcal{T}}) \ge \gamma(\mathcal{T}) - 2\|\mathcal{T}\|\delta\left(R(\breve{\mathcal{T}}), R(\mathcal{T})\right) - \|\mathcal{S}\| \tag{2.24}$$

*and*

$$\gamma(\breve{\mathcal{T}}) \ge \gamma(\mathcal{T}) - 2\|\mathcal{T}\|\delta\left(N(\breve{\mathcal{T}}), N(\mathcal{T})\right) - \|\mathcal{S}\|. \tag{2.25}$$

*Proof.* To prove (2.24), we first note that $D(\mathcal{S}^*) \supset D(\mathcal{T}^*)$ and $\mathcal{S}^*(0) \subset \mathcal{T}^*(0)$ since $D(\mathcal{S}^*) = \mathcal{S}(0)^\perp$, $D(\mathcal{T}^*) = \mathcal{T}(0)^\perp$ and $\mathcal{S}^*(0) = D(\mathcal{S})^\perp$, $\mathcal{T}^*(0) = D(\mathcal{T})^\perp$. Let $\mathscr{D}(\mathcal{T}) := D(\mathcal{T}) \cap S_{N(\mathcal{T})^\perp}$ and let $z \in \mathscr{D}(\breve{\mathcal{T}}^*)$. Then given $\varepsilon > 0$, there exists $w \in \mathscr{D}(\mathcal{T}^*)$ such that $\|z - w\| \leq \operatorname{dist}(z, \mathscr{D}(\mathcal{T}^*)) + \varepsilon$. Note that $\mathscr{D}(\mathcal{T}^*) := D(\mathcal{T}^*) \cap S_{N(\mathcal{T}^*)^\perp}$ is nonempty since $0 < \gamma(\mathcal{T}) = \gamma(\mathcal{T}^*) < \infty$ and so $D(\mathcal{T}^*) \not\subset N(\mathcal{T}^*)$. Hence

$$\begin{aligned}
\|\mathcal{T}^* z\| &= \|\mathcal{T}^*(w + z - w)\| = \|\mathcal{T}^* w + \mathcal{T}^*(z - w)\| \\
&\geq \|\mathcal{T}^* w\| - \|\mathcal{T}^*(z - w)\| \geq \gamma(\mathcal{T}) - \|\mathcal{T}^*(z - w)\| \\
&\geq \gamma(\mathcal{T}) - \|\mathcal{T}^*\| \|(z - w)\| \geq \gamma(\mathcal{T}) - \|\mathcal{T}^*\| \left[\operatorname{dist}(z, \mathscr{D}(\mathcal{T}^*)) + \varepsilon\right]
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\begin{aligned}
\|\mathcal{T}^* z\| &\geq \gamma(\mathcal{T}) - \|\mathcal{T}^*\| \left[\operatorname{dist}(z, \mathscr{D}(\mathcal{T}^*))\right] \\
&\geq \gamma(\mathcal{T}) - \|\mathcal{T}^*\| \sup_{z \in \mathscr{D}(\breve{\mathcal{T}}^*)} \{\operatorname{dist}(z, \mathscr{D}(\mathcal{T}^*))\} \\
&\geq \gamma(\mathcal{T}) - \|\mathcal{T}^*\| \, \widehat{\delta}_{\breve{\mathcal{T}}^*} \left(N(\breve{\mathcal{T}}^*)^\perp, N(\mathcal{T}^*)^\perp\right), \text{ since } D(\breve{\mathcal{T}}^*) = D(\mathcal{T}^*) \\
&\geq \gamma(\mathcal{T}) - \|\mathcal{T}^*\| \, \widehat{\delta}_{\breve{\mathcal{T}}^*} \left(N(\breve{\mathcal{T}}^*)^\perp, N(\mathcal{T}^*)^\perp\right) \\
&\geq \gamma(\mathcal{T}) - 2\|\mathcal{T}^*\| \, \delta \left(N(\breve{\mathcal{T}}^*)^\perp, N(\mathcal{T}^*)^\perp\right) \\
&= \gamma(\mathcal{T}) - 2\|\mathcal{T}^*\| \, \delta \left(R(\breve{\mathcal{T}}), R(\mathcal{T})\right) \qquad (2.26)
\end{aligned}$$

and therefore

$$\begin{aligned}
\gamma(\breve{\mathcal{T}}) &= \gamma(\breve{\mathcal{T}}^*) = \inf_{z \in \mathscr{D}(\breve{\mathcal{T}}^*)} \left\|\breve{\mathcal{T}}^* z\right\| = \inf_{z \in \mathscr{D}(\breve{\mathcal{T}}^*)} \|\mathcal{T}^* z + \mathcal{S}^* z\| \\
&\geq \inf_{z \in \mathscr{D}(\breve{\mathcal{T}}^*)} \{\|\mathcal{T}^* z\| - \|\mathcal{S}^* z\|\} \geq \inf_{z \in \mathscr{D}(\breve{\mathcal{T}}^*)} \{\|\mathcal{T}^* z\|\} - \|\mathcal{S}^*\| \\
&\geq \gamma(\mathcal{T}) - 2\|\mathcal{T}^*\| \, \delta \left(R(\breve{\mathcal{T}}), R(\mathcal{T})\right) \text{ by (2.26)} \\
&= \gamma(\mathcal{T}) - 2\|\mathcal{T}\| \, \delta \left(R(\breve{\mathcal{T}}), R(\mathcal{T})\right).
\end{aligned}$$

Inequality (2.25) can be proved in a similar way with the help of the equality $\delta(M, N) = \delta\left(M^\perp, N^\perp\right)$ [4, IV Theorem 2.9]. □

Let $\mathcal{S}, \mathcal{T} \in CLR(H_1, H_2)$ where $H_1$ and $H_2$ are Hilbert spaces and consider the perturbation $\breve{\mathcal{T}} = \mathcal{T} + \mathcal{S}$ of $\mathcal{T}$ by $\mathcal{S}$. We consider some norm bounds for the generalized inverse $\breve{\mathcal{T}}^\dagger$ of $\breve{\mathcal{T}}$. For ease of notation, we let $\delta_N = \delta\left(N(\breve{\mathcal{T}}), N(\mathcal{T})\right)$ and $\delta_R = \delta\left(R(\breve{\mathcal{T}}), R(\mathcal{T})\right)$.

**Theorem 2.10.** *Let $H_1$ and $H_2$ be Hilbert spaces and let $\mathcal{S}, \mathcal{T} \in CLR(H_1, H_2)$ be such $0 < \gamma(\mathcal{T}) < \infty$ , $\dim N(\mathcal{T}) < \infty$, $D(\mathcal{S}) \supset D(\mathcal{T})$, $\mathcal{S}(0) \subset \mathcal{T}(0)$, and $\|\mathcal{S}\| < \gamma(\mathcal{T})$. Consider the perturbation $\breve{\mathcal{T}} = \mathcal{T} + \mathcal{S}$ of $\mathcal{T}$ by $\mathcal{S}$.*

*(i) If* $2\delta_R\|\mathcal{T}\| + \|\mathcal{S}\| < \left\|\mathcal{T}^{\dagger}\right\|^{-1}$ *then* $\|\breve{\mathcal{T}}^{\dagger}\| \leq \dfrac{\left\|\mathcal{T}^{\dagger}\right\|}{1 - \|\mathcal{T}^{\dagger}\| \left(2\delta_R\|\mathcal{T}\| + \|\mathcal{S}\|\right)}$.

*(ii) If* $2\delta_N\|\mathcal{T}\| + \|\mathcal{S}\| < \left\|\mathcal{T}^{\dagger}\right\|^{-1}$ *then* $\|\breve{\mathcal{T}}^{\dagger}\| \leq \dfrac{\left\|\mathcal{T}^{\dagger}\right\|}{1 - \|\mathcal{T}^{\dagger}\| \left(2\delta_N\|\mathcal{T}\| + \|\mathcal{S}\|\right)}$.

*(ii) If either* $N(\breve{\mathcal{T}}) \subseteq N(\mathcal{T})$ *or* $R(\breve{\mathcal{T}}) \subseteq R(\mathcal{T})$ *then* $\|\breve{\mathcal{T}}^{\dagger}\| \leq \dfrac{\left\|\mathcal{T}^{\dagger}\right\|}{1 - \|\mathcal{T}^{\dagger}\| \, \|\mathcal{S}\|}$.

*Proof.* First we note that $\breve{\mathcal{T}}$ has closed range [5, Theorem 32] and therefore the generalized inverse $\breve{\mathcal{T}}$ as defined by (2.3) exists. The results follow by noting that $\left\|\breve{\mathcal{T}}^{\dagger}\right\| = \gamma\left(\breve{\mathcal{T}}\right)^{-1}$ and $\left\|\mathcal{T}^{\dagger}\right\| = \gamma\left(\mathcal{T}\right)^{-1}$ and using (2.24) to get $(i)$ and (2.25) to get $(ii)$. Lastly, inequality $(iii)$ follows from either $(i)$ or $(ii)$ by noting that if $N(\breve{\mathcal{T}}) \subseteq N(\mathcal{T})$ then $\delta_N = 0$ and if $R(\breve{\mathcal{T}}) \subseteq R(\mathcal{T})$ then $\delta_R = 0$ and that $\|\mathcal{S}\| < \gamma(\mathcal{T}) = \left\|\mathcal{T}^{\dagger}\right\|^{-1}$ by hypothesis. □

# 3 Conclusion

In this paper, we devoted out attention develop the theory of generalized inverses of linear relations, paying particular attention to linearity and boundedness. Some of our results are generalizations of similar results in the single valued operator case. Many times one encounters problems in differential equations and many areas of analysis where linear relations form a basic tool for developing the necessary theory to study such a problem. For example, the study of the spectral theory of non-densely defined ordinary differential operators relies heavily on the theory of linear relations and their adjoints. Our next focuss is going to be on applying the theory developed here to solve some problem in other areas mathematics and related fields.

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