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SIR Models with Fuzzy Initial Conditions

I.K. Youssef [1], M. Khalifa Saad[1], Ali Dumlu[1], Somia A. Asklany[2]

[1]Department of Mathematics
Faculty of Science
Islamic University of Madinah
Saudi Arabia

[2]Computers and IT Department
Turif, Northern Border University
Saudi Arabia

email: kaoud22@hotmail.com, mohammed.khalifa@iu.edu.sa,
somiaasklany@hotmail.com

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Abstract

The SIR model is a mathematical model of epidemics involving the unknowns Susceptible, Infected, and Recovered. We consider a numerical solution of the Fuzzy SIR model (FSIR) with fuzzy initial conditions. Actual values of the parameters are used for the Kingdom of Saudi Arabia region for the peak period of July-August 2020 for COVID-19. Due to the nonlinearities in the SIR model and the included fuzzy concepts, the solution methods are focused on numerical techniques. The Euler method is the simplest numerical technique with minimum function evaluation per step which is an effective factor in a fuzzy environment. The results show the effective impact of lockdown policies. Moreover, the fuzzy concepts give global attitudes about the components of the SIR model. Furthermore, we establish a graphical representation of the actual documented data with the solution of the FSIR model for the data of our sample for different values of the fuzzy parameter.

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1 Introduction

Classical mathematical models describing many existing phenomena in different fields of life incorporate a high degree of subjectivity in their developments. Moreover, some specific mathematical models involving vagueness, imprecision, and uncertainty need treatments considering these factors. Using the concepts of fuzzy mathematics offer explanations for some faults and enhances the understanding of the behaviors of many realistic models. Recently, extensive research was introduced to consider the spread of epidemic diseases [1, 2]. Many researchers considered the SIR model for different regions [3, 4]. The SIR model is a system of three coupled nonlinear ordinary differential equations for the unknowns: $S(t)$ (Susceptible), $I(t)$ (Infected), and $R(t)$ (Recovered), with $N = S(t) + I(t) + R(t)$ being a fixed closed population size under consideration (Sample Size) [1, 2, 3, 4, 5, 6].

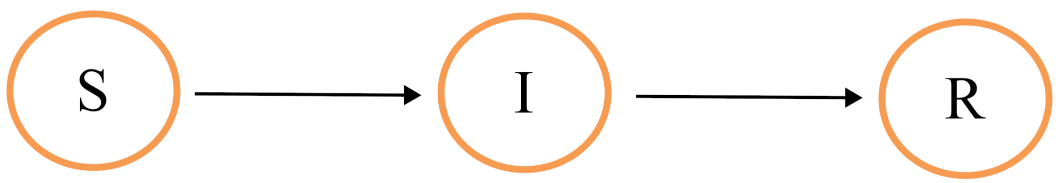

Figure 1: Flow direction between the components of the sample study.

The SIR model was established by W.O. Kermack and A.G. McKendrick in a set of three articles from 1927, 1932, and 1933. The model is based on the bilinear incidence rate other dynamics can be considered in subsequent works.

$$\begin{aligned} & S'(t) = -\beta S I, \\ & I'(t) = \beta S I - \alpha I, \\ & R'(t) = \alpha I. \\ & S(t_0) = S_0, I(t_0) = I_0, R(t_0) = R_0. \end{aligned} \tag{1.1}$$

In the classical treatment of the SIR model, the initial conditions $S(t_0) = S_0, I(t_0) = I_0, R(t_0) = R_0; t_0$ were taken at any point for which reliable data exists. Also, the SIR model involves two additional nonnegative parameters: α and β; The parameter α is known as the recovery coefficient and the parameter β is the transmission coefficient. The solutions depend on the parameter values as well as the initial conditions. In most epidemics, it is difficult to determine exactly how many new infections there are each day since only those that are removed can be counted accurately. Sometimes, the

symptoms of the infection take time to appear. In [5, 6], we considered implementing the SIR model for the Kingdom of Saudi Arabia from April 2020 to December 2020 (the period of COVID-19 without medical treatments). We considered a sample of study $S(t_0) = 400000$ cases and we determined the model parameters. In this present work, we extend our study to the fuzzy environment and the initial values are taken as fuzzy numbers with their centers (peak of the triangular fuzzy number) at the documented data values [7]. Therefore, the solutions will be fuzzy functions giving bounds for the expected crisp solution values.
In the classical solution, $R(t)$ must be an increasing function and $S(t)$ must be a decreasing function. So the fuzzy solution keeps this attitude for fixed values of the fuzzy parameter r. Solution methods for such nonlinear systems are directed towards numerical techniques; the complications are at least doubled for fuzzy environments. So numerical techniques are still the most appropriate solution methods for such systems. Despite its low accuracy, the Euler method is the simplest to use among all numerical techniques for solving systems of differential equations [8]. There are still a limited number of publications in which the numerical solution of nonlinear initial value problems in a fuzzy environment is considered [8, 9, 10].

1.1 Triangular Fuzzy Numbers

A fuzzy number $\tilde{v}(r)$ is a convex normalized fuzzy set of the real line $\mathcal{R}$ such that: $\left\{\mu_{\tilde{V}}(x) : R \to [0,1], \forall x \in R\right\}$, the set of all fuzzy numbers on $\mathcal{R}$ is known as $\mathcal{R}_F$. We consider triangular fuzzy numbers in two forms with the the first being $v = (a, b, c)$ and the second being the parametric form. A fuzzy number v in a parametric structure is described as an ordered pair $(\underline{v}, \bar{v})$ with the first component $\underline{v}(r)$ and the second component $\bar{v}(r), 0 \leq r \leq 1$ satisfying the following requirements [8, 9, 10, 11, 12]: $\underline{v}(r)$ is a bounded monotonic non-decreasing left-continuous function over $[0, 1]$, $\bar{v}(r)$ is a bounded monotonic non-increasing left-continuous function over $[0, 1]$. $\underline{v}(r) \leq \bar{v}(r), 0 \leq \mathrm{r} \leq 1$. $\tilde{v}(r) = (\underline{v}, \bar{v}) = (a + r(b - a), c - r(c - b))$, as shown in Fig. 2.

It is accepted that the fuzzy numbers $\tilde{u}(r) = (\underline{u}(r), \bar{u}(\mathrm{r})), \tilde{v}(r) = (\underline{v}(r), \bar{v}(\mathrm{r}))$ and scalar k satisfy the following fuzzy arithmetic:

1. $\tilde{u} = \tilde{v}$ if and only if $\underline{u}(r) = \underline{v}(r)$ and $\bar{u}(r) = \bar{v}(r)$.
2. $\tilde{u} + \tilde{v} = (\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r))$.
3. $k\tilde{v} = \begin{cases} (k\underline{v}(r), k\bar{v}(r)), & k \geq 0, \\ (k\bar{v}(r), k\underline{v}(r)), & k < 0, \end{cases}$
4. $\tilde{u} \cdot \tilde{v} = (\underline{w}(r), \bar{w}(r))$;

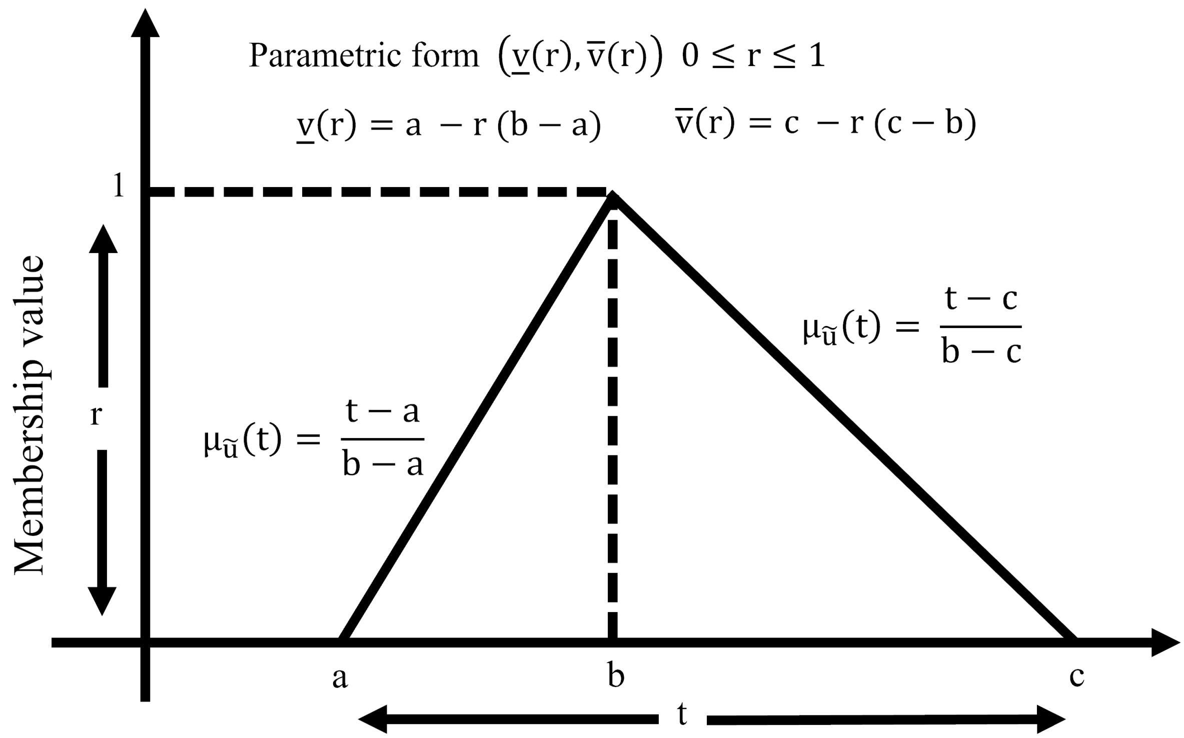

Figure 2: Different representations of triangular fuzzy number.

$$\underline{w}(r) = \min\{\underline{u}\underline{v}, \underline{u}\bar{v}, \bar{u}\underline{v}, \bar{u}\bar{v}\}$$
$$\bar{w}(\mathrm{r}) = \max\{\underline{u}\underline{v}, \underline{u}\bar{v}, \bar{u}\underline{v}, \bar{u}\bar{v}\}$$

Also, it is generally accepted that any function is a fuzzy function if at least its range assigns fuzzy numbers. We are interested in fuzzy functions of the form:

$$\begin{aligned} &f : (a,b) \to \mathcal{R}_{\mathcal{F}}, (a,b) \subset R. \\ &\tilde{f}(t,r) = [\underline{f}(t,r), \bar{f}(t,r)]. \end{aligned} \tag{1.2}$$

whose derivative take the form:

$$\tilde{f}'(t,r) = \left[\underline{f}'(t,r), \bar{f}'(t,r)\right], \tag{1.3}$$

2 Fuzzy Initial Values

The Fuzzy Initial Value Problem (FIVP) is written in the form:

$$\begin{aligned} &\check{y}'(t,r) = \tilde{f}(t,\check{y}(t,r)), \quad a \le t \le b, \\ &\check{y}(a,r) = y_0(r), \end{aligned} \tag{2.1}$$

with

$$\left[\tilde{f}(t,\check{y}(t,r)\right]^r = \tilde{f}\left(t,\left[\underline{y_r},\overline{y_r}\right]\right) = \left[\min f\left(t,\left[\underline{y_r},\overline{y_r}\right]\right),\max f\left(t,\left[\underline{y_r},\overline{y_r}\right]\right)\right]$$

$y_0 \in \mathcal{R}_F$ and $f : [a,b] \times \mathcal{R}_F \to \mathcal{R}_F$ is a fuzzy continuous function obtained by applying Zadeh's extension principle to the real function $f : [a,b] \times \mathcal{R}_F \to \mathcal{R}_F$. Thus the FIVP is associated with a system of two crisp initial value problems of the form:

$$\begin{aligned} &\underline{y'}(t,r) = \underline{f}(t,\underline{y}(t,r),\bar{y}(t,r)), \underline{y}(a,r) = \underline{y_0}(r) \\ &\overline{y'}(t,r) = \bar{f}(t,\underline{y}(t,r),\bar{y}(t,r)), \bar{y}(a,r) = \overline{y_0}(r) \end{aligned} \tag{2.2}$$

When solving initial value problems by numerical techniques like Euler method, the continuous domain $[a,b]$ is replaced by a discreate set of equally spaced grid points with grid spacing $h, t_n = a + nh, y\left(t_n\right) = y_n$. Accordingly, the fuzzy Euler method for this system is written as:

$$\begin{aligned} &\underline{y}_{n+1} = \underline{y}_n + k_{11}, \\ &\bar{y}_{n+1} = \bar{y}_n + k_{21}, n = 0,1,2,\cdots,N, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} k_{11} &= \text{Min}\left\{h\underline{f}\left(t_n,\underline{y}_n,\bar{y}_n\right), h\bar{f}\left(t_n,\underline{y}_n,\bar{y}_n\right)\right\}, \\ k_{21} &= \text{Max}\left\{h\underline{f}\left(t_n,\underline{y}_n,\bar{y}_n\right), h\bar{f}\left(t_n,\underline{y}_n,\bar{y}_n\right)\right\}. \end{aligned} \tag{2.4}$$

3 The FSIR Model

The FSIR model can be written in the form:

$$\begin{aligned} &\check{S}'(t,r) = -\beta\check{S}\check{I}, \\ &\check{I}'(t,r) = \beta\check{S}\check{I} - \alpha\check{I}, \\ &\check{R}'(t,r) = \alpha\check{I}, \\ &\check{S}\left(t_0,r\right) = \left[S_L + r\left(S_M - S_L\right), S_R - r\left(S_R - S_M\right)\right], \\ &\check{I}\left(t_0,r\right) = \left[I_L + r\left(I_M - I_L\right), I_R - r\left(I_R - I_M\right)\right], \\ &\check{R}\left(t_0,r\right) = \left[R_L + r\left(R_M - R_L\right), R_R - r\left(R_R - R_M\right)\right], \end{aligned} \tag{3.1}$$

where the suffix L is used for the left of the triangular fuzzy number or lower bound (denoted by a in Fig. 2), the suffix M is used for the modal value of the triangular fuzzy number (denoted by b in Fig. 2) and the suffix R is used for the right of the triangular fuzzy number or upper bound (denoted by c in Fig. 2). The solution of the FSIR model is obtained through the solution of the associated crisp system consisting of six equations as illustrated in (2.3), solutions of the FSIR model (3.1) give the unknowns as fuzzy functions which generalize the solution of the crisp SIR model (1.1).

4 Results

The results of solving the FSIR system (3.1) with the extension of the fuzzy Euler method (2.3) and (2.4) to a system of six equations with the initial values as summarized in Figures 3 and 4 compared with the documented data [7] and the parameters of the model as determined in [5, 6] are given for the peak period July-August in Figures 3 and 4. The model values of the fuzzy numbers corresponding to the initial conditions are taken at the documented values. The solution of the classical model (1.1) can be obtained as a special case ($r = 1$). The results shown in figures 3 and 4 give an overestimation to the documented data due to the best application of the lockdown measures in the Kingdom of Saudi Arabia during the considered periods. Using fuzzy models is an effective approach because of its global insights. Our results are highly acceptable in comparison to the documented data and the application of highly effective safety polices. It is recommended to use fuzzy models for epidemiological studies due to the inclusion of the extension principles which guarantee the inclusion of classical solutions.

5 Discussion

Most of the mathematical models which can be used to predict the behaviors of a natural phenomenon, especially those concerning the spread of epidemic diseases appear as systems of differential equations [1, 2, 3, 4]. Solutions of such classical systems suffer from vagueness, imprecision, and uncertainty which appear in the data of the problem. Therefore, planning methods for controlling the performances of such a phenomenon depends on solutions for such systems under different initial conditions. The SIR model is based on considering the rate of change of the size of its components, the bilinear incidence rate between its components $(S(t), I(t) and R(t))$ as shown in (1.1)

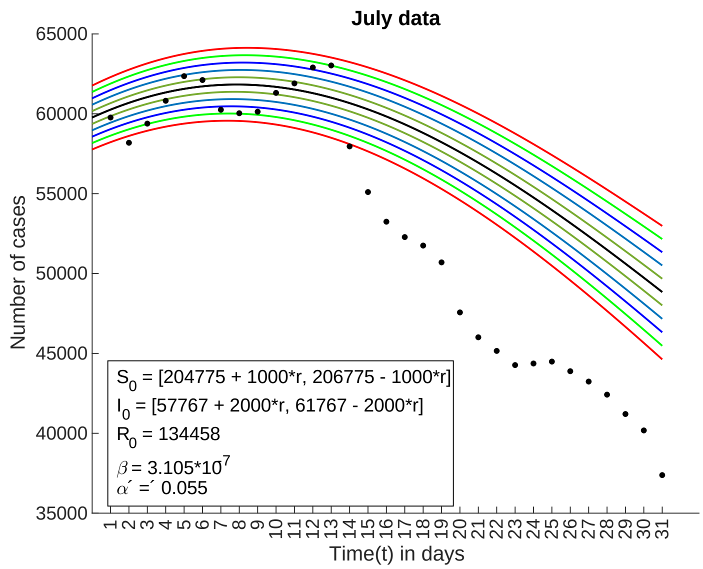

Figure 3: Documented daily active infected cases (July 2020) versus the fuzzy Euler soltion for different values of the fuzzy parameter.

so the SIR model is a nonlinear system of differential equations such systems are solved by numerical techniques. The situation for nonlinear systems of differential equations in fuzzy domain has an extra complication in order to keep the correct formats of the resultant solutions as appropriate fuzzy functions and accordingly keeping the decrease in $S(t, r)$ for fixed values of r, and the increase in $R(t, r)$, see Fig. 3. The strength of fuzzy differential equations lies in that their solutions cover a wide range of expected solution values especially when the initial values are not precisely defined Fig. 3 and Fig. 4. Solving fuzzy differential equations concerning some problem gives an overview of the behavior of the solution of the same problem for different attitudes. Our results here generalize our findings in [1, 2, 3, 4]. although in [1, 2, 3, 4] we used more accurate techniques fourth order Runge-Kutta method and different forms of Picard iteration techniques, which will be our concern in the next work.

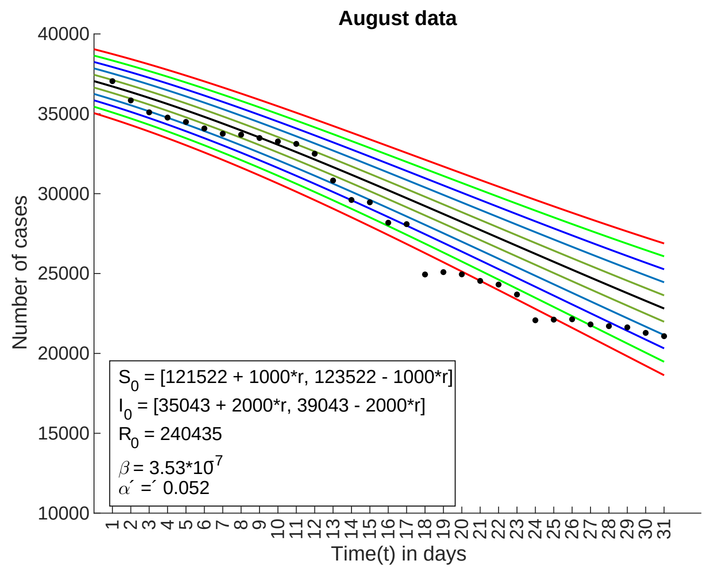

Figure 4: Documented daily active infected cases (August 2020) versus the fuzzy Euler soltion for different values of the fuzzy parameter.

6 Conclusion

The FSIR model is still an effective simple mathematical models which can help policy makers or strategic planers to obtain a quick overview. Application of mathematical epidemic models should be restricted to short time scales and should consider the accuracy of the documented data. The fuzzy treatments are promising tracks in handling epidemic models. The overestimations appear in the solutions due to the effective lockdown polices. Despite its low accuracy, Euler method yields quick acceptable results over small domains. We look forward to applying more accurate methods over large domains.

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