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On Centrally-Extended Multiplicative (generalized)-(α,β)-Reverse Derivations in Semiprime Rings

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Abstract

The aim of this paper is to examine the behavior of centrally- extended multiplicative (generalized)-(α,β)-reverse derivations by investigating various algebraic identities. These include $T(\mathfrak{m}\mathfrak{n}) \mp \beta(\mathfrak{n})H(\mathfrak{m}) \in Z$, $T(\mathfrak{m}\mathfrak{n}) \mp g(\mathfrak{n})\alpha(m) \in Z$, and $T(\mathfrak{m}\mathfrak{n}) \mp g(\mathfrak{m})\alpha(\mathfrak{n}) \in Z$ for any elements $\mathfrak{m}, \mathfrak{n}$ belonging to specific subsets of $\mathbb{R}$.

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1 Introduction

Let $Z(\mathbb{R})$ denote the center of a ring $\mathbb{R}$ and let α, β be mappings of $\mathbb{R}$. For any e and f belonging to the ring $\mathbb{R}$, the commutator $ef - fe$ is denoted by $[e, f]$ [1]. A ring $\mathbb{R}$ is considered semiprime if, whenever $e\mathbb{R}e = 0$, we have $e = 0$. A derivation D (for brevity) is an additive mapping defined for elements a and e in the ring $\mathbb{R}$, satisfying $d(ae) = d(a)e + ad(e)$. An additive mapping $\mathcal{F}$ of $\mathbb{R}$ is called a generalized derivation GD (for brevity) associated with d if there exists a D mapping d of $\mathbb{R}$ such that $\mathcal{F}(ae) = \mathcal{F}(a)e + ad(e)$, for all the elements a and e of $\mathbb{R}$ [2]. Several mathematicians have developed and expanded this concept in different ways.

One of these extensions is the centrally-extended derivations CE-D (for brevity), introduced by Bell and Daif [3], who considered a mapping $v : \mathbb{R} \to \mathbb{R}$ such that $v(\mathfrak{m} + \mathfrak{n}) - v(\mathfrak{m}) - v(\mathfrak{n}) \in Z$ and $v(\mathfrak{m}\mathfrak{n}) - v(\mathfrak{m})\mathfrak{n} - \mathfrak{m}v(\mathfrak{n}) \in Z$ is true, for all the elements $\mathfrak{m}$ and $\mathfrak{n}$ of $\mathbb{R}$.
Moreover, in [4], this notion has been developed to centrally-extended (α,β)-derivation CE-(α,β)-D (for brevity) containing other types of maps. A CE-(α,β)-D is a mapping v of $\mathbb{R}$ that satisfies $v(\mathfrak{m} + \mathfrak{n}) - v(\mathfrak{m}) - v(\mathfrak{n}) \in Z$ and $v(\mathfrak{m}\mathfrak{n}) - v(\mathfrak{m})\alpha(\mathfrak{n}) - \beta(\mathfrak{m})v(\mathfrak{n}) \in Z$ holds for every $\mathfrak{m}, \mathfrak{n} \in \mathbb{R}$. Furthermore, any map T of R that satisfies the conditions $T(\mathfrak{m}+\mathfrak{n})-T(\mathfrak{m})-T(\mathfrak{n}) \in Z$ and $T(\mathfrak{m}\mathfrak{n})-T(\mathfrak{m})\alpha(\mathfrak{n})-\beta(\mathfrak{m})g(\mathfrak{n}) \in Z$ is called a centrally-extended generalized (α,β)-derivation CE(g)-(α,β)-D (for brevity), where g is a CE-(α,β)-D of $\mathbb{R}$.

In 1991, Daif [5] introduced the notion of a multiplicative derivation which, unlike conventional derivations, does not necessarily need to be additive. In addition, Daif and El-Sayiad [6], considering the concept of G mapping, extended this concept by introducing the notion of generalized multiplicative derivation. There are numerous findings regarding this subject [7, 8, 9].

Herstein [10] initially introduced the concept of reverse derivation RD (for brevity). An additive mapping v from $\mathbb{R}$ to itself is called an RD mapping if it is additive and satisfies the condition $v(\mathfrak{m}\mathfrak{n}) = v(\mathfrak{n})\mathfrak{m} + \mathfrak{n}v(\mathfrak{m})$ for all elements $\mathfrak{m}, \mathfrak{n} \in \mathbb{R}$. Many authors studied this concept, in many directions, like [11]. Moreover, this concept has undergone considerable development, with generalizations being presented as follows [12]:
If a mapping T of $\mathbb{R}$ such that for all $\mathfrak{m}, \mathfrak{n} \in \mathbb{R}$, the condition $T(\mathfrak{m}\mathfrak{n}) = T(\mathfrak{n})\mathfrak{m} + \mathfrak{n}g(\mathfrak{m})$ holds, then T is called multiplicative(generalized)-reverse derivation M(g)-RD (for brevity), where g is any map. Subsequently, those authors expanded the concept of a M(g)-RD mapping, leading to a multiplicative (generalized) (α,β)-reverse derivation M(g)-(α,β)-RD, (for brevity).

A mapping T of $\mathbb{R}$ is MG-(α,β)-RD if for all $\mathfrak{m}, \mathfrak{n} \in \mathbb{R}$, $T(\mathfrak{m}\mathfrak{n}) = T(\mathfrak{n})\alpha(\mathfrak{m}) + \beta(\mathfrak{n})g(\mathfrak{m})$ holds, where g is a mapping. Many results can be found in [13, 14, 15].

Muthana and AlKhamisi[16] defined a map T of $\mathbb{R}$ as a centrally-extended multiplicative(generalized)-(α, β) derivation for every $\mathfrak{m}, \mathfrak{n} \in \mathbb{R}$, $T(\mathfrak{m}\mathfrak{n}) - T(\mathfrak{m})\alpha(\mathfrak{n}) - \beta(\mathfrak{m})g(\mathfrak{n}) \in Z$ hold, where g is mapping for $\mathbb{R}$ CEM(g)(α,β)D, (for brevity). Moreover, they investigated the identities:
$\mathbb{F}(ef) \pm \beta(e)\mathbb{G}(f) \in Z$, $\mathbb{F}(ef) \pm g(e)\alpha(f) \in Z$, and $\mathbb{F}(ef) \pm g(f)\alpha(e) \in Z$, for all $e, f \in Z$.

In this paper, we study the above results by presenting the new concept, called centrally-extended multiplicative (generalized)-(α,β)-reverse derivation $\mathbb{CEM}(\eth) - (\alpha, \beta) - RD$ (for brevity). Additionally, we discuss several identities on a suitable subset of a semiprime ring such as $T(\mathfrak{m}\mathfrak{n}) \mp \beta(\mathfrak{n})H(\mathfrak{m}) \in Z$, $T(\mathfrak{m}\mathfrak{n}) \mp g(\mathfrak{n})\alpha(\mathfrak{m}) \in Z$, and $T(\mathfrak{m}\mathfrak{n}) \mp g(\mathfrak{m})\alpha(\mathfrak{n}) \in Z$, where α, β are mappings of $\mathbb{R}$, and H is M(g)-(α,β)-RD.

2 Preliminaries

The starting point will be to introduce the definition of $\mathbb{CEM}(\eth) - (\alpha, \beta) - RD$, (for brevity) as follows:

Definition 2.1. *A mapping T of $\mathbb{R}$ is said to be $\mathbb{CEM}(\eth) - (\alpha, \beta) - RD$ associated with a map d if the condition $T(\mathfrak{m}\mathfrak{n}) - T(\mathfrak{n})\alpha(\mathfrak{m}) - \beta(\mathfrak{n})d(\mathfrak{m}) \in Z$ holds for every $\mathfrak{m}, \mathfrak{n} \in \mathbb{R}$ where α, β are mappings from R.*

Throughout this work, $\mathbb{K}$ is a symbol of the left ideal of $\mathbb{R}$.

3 Main results

Theorem 3.1. *For an anti-homomorphism β , α a mapping on $\mathbb{K}$ and a $\mathbb{CEM}(\eth) - (\alpha, \beta) - RD$ mapping T of $\mathbb{R}$ associated to d, for all element $\mathfrak{m} \in \mathbb{K}$, the statement $K[(d \mp g)(\mathfrak{m})] = 0$ is true whenever $T(\mathfrak{m}\mathfrak{n}) \mp \beta(\mathfrak{n})H(\mathfrak{m})$ is contained in Z, H is M(g)-(α, β)-RD mapping of $\mathbb{R}$ associated to g and $\alpha(\mathbb{K})$ contained in $\mathbb{K}$, $\beta(\mathbb{K}) = \mathbb{K}$.*

Proof.

$$T(\mathfrak{m}\mathfrak{n}) \mp \beta(\mathfrak{n})H(\mathfrak{m}) \in Z. \tag{3.1}$$

Replacing $\mathfrak{m}$ by $\mathfrak{m}\mathbf{z}$ in (3.1), we get

$$T(\mathfrak{m}\mathbf{z}\mathfrak{n}) \mp \beta(\mathfrak{n})H(\mathfrak{m}\mathfrak{n}) \in Z.$$

$$(T(\mathbf{z}\mathfrak{n}) \mp \beta(\mathfrak{n})H(\mathbf{z}))\alpha(\mathfrak{m}) + \beta(\mathbf{z}\mathfrak{n})d(\mathfrak{m}) \mp \beta(\mathfrak{n})\beta(\mathbf{z})g(\mathfrak{m}) \in Z.$$

By using (3.1), we have

$$\beta(\mathfrak{n})\beta(\mathbf{z})d(\mathfrak{m}) \mp \beta(\mathfrak{n})\beta(\mathbf{z})g(\mathfrak{m}) \in Z. \tag{3.2}$$

Commuting (3.2) with $\alpha(\mathfrak{m})$, we get

$$[\beta(\mathfrak{n})\beta(\mathbf{z})(d \mp g)(\mathfrak{m}), \alpha(\mathfrak{m})] = 0.$$

Since $\beta(\mathbb{K}) = \mathbb{K}$, $[\mathfrak{n}\mathbf{z}(d \mp g)(\mathfrak{m}), \alpha(\mathfrak{m})] = 0$.

By [[16], Lemma 2.1], we get

$$\mathbb{K}[(d \mp g)(\mathfrak{m}), \alpha(\mathfrak{m})] = 0.$$

By semiprimeness of $\mathbb{K}$, if $\mathbb{K}$ is an ideal of $\mathbb{R}$, then get

$$[(d \mp g)(\mathfrak{m}), \alpha(\mathfrak{m})] = 0.$$

Theorem 3.2. *For a surjective mapping β of $\mathbb{K}$ and a $\mathbb{CEM}(\eth)-(\alpha,\beta)-RD$ mapping T of $\mathbb{R}$ associated to d, the statement $K[(d(\mathfrak{m}), \alpha(\mathfrak{m})] = 0$ is true for all element $\mathfrak{m} \in \mathbb{K}$ whenever $T(\mathfrak{m}\mathfrak{n}) \mp g(\mathfrak{n})\alpha(\mathfrak{m}) \in Z$ and $\alpha(\mathbb{K}) \subseteq \mathbb{K}$, α acts anti-homomorphism and β homomorphism on $\mathbb{K}$.*

Proof.

$$T(\mathfrak{m}\mathfrak{n}) \mp g(\mathfrak{n})\alpha(\mathfrak{m}) \in Z. \tag{3.3}$$

If we replace $\mathfrak{m}$ by $\mathfrak{m}\mathbf{z}$ in (3.3), then

$$T(\mathfrak{m}\mathbf{z}\mathfrak{n}) \mp g(\mathfrak{n})\alpha(\mathfrak{m}\mathbf{z}) \in Z.$$

Since T is $\mathbb{CEM}(\eth) - (\alpha,\beta) - RD$ of $\mathbb{R}$, and α is anti-homomorphism, we have

$$(T(\mathbf{z}\mathfrak{n}) \mp g(\mathfrak{n})\alpha(\mathbf{z}))\alpha(\mathfrak{m}) + \beta(\mathbf{z}\mathfrak{n})d(\mathfrak{m}) \in Z.$$

Using the hypothesis, and since β is homomorphism, we get

$$\beta(\mathbf{z})\beta(\mathfrak{n})d(\mathfrak{m}) \in Z. \tag{3.4}$$

Commuting (3.4) with $\alpha(\mathfrak{m})$, we obtain

$$[\beta(\mathbf{z})\beta(\mathfrak{n})d(\mathfrak{m}), \alpha(\mathfrak{m})] = 0.$$

Since $\beta(\mathbb{K}) = \mathbb{K}$,

$$[\mathbf{z}\mathfrak{n}\mathbf{d}(\mathfrak{m}), \alpha(\mathbf{m})] = 0.$$

By [[16], Lemma 2.1], we get

$$\mathbb{K}[d(\mathfrak{m}), \alpha(\mathfrak{m})] = 0.$$

Theorem 3.3. *For a surjective mapping α on $\mathbb{K}$ and a $\mathbb{CEM}(\eth) - (\alpha, \alpha) - RD$ mapping T of $\mathbb{R}$ associated to d, for all the elements $\mathfrak{m}$ and $\mathfrak{n}$ of $\mathbb{K}$ the statement $K[(d(\mathfrak{m}), \alpha(\mathfrak{m})] = 0$ holds whenever $T(\mathfrak{m}\mathfrak{n}) \mp g(\mathfrak{m})\alpha(\mathfrak{n}) \in Z$, where α acts homomorphism on $\mathbb{K}$.*

Proof.

$$T(\mathfrak{m}\mathfrak{n}) \pm g(\mathfrak{m})\alpha(\mathfrak{n}) \in z. \tag{3.5}$$

Replacing $\mathfrak{m}$ with $\mathfrak{m}\ \mathbf{z}$ in (3.5), we get

$$T(\mathbf{z}\mathfrak{n})\alpha(\mathfrak{m}) + \alpha(\mathbf{z}\mathfrak{n})d(\mathfrak{m}) \pm g(\mathfrak{m}\mathbf{z})\alpha(\mathfrak{n}) \in z. \tag{3.6}$$

Now, taking $\mathbf{z}$ instead of $\mathfrak{m}$ in (3.5), we have

$$T(\mathbf{z}\mathfrak{n}) \pm g(\mathbf{z})\alpha(\mathfrak{n}) \in z. \tag{3.7}$$

Commuting (3.7) with $\alpha(\mathfrak{m})$, we get

$$[T(\mathbf{z}\mathfrak{n}) \pm g(\mathbf{z})\alpha(\mathfrak{n}), \alpha(\mathfrak{m})] = 0.$$

$$[T(\mathbf{z}\mathfrak{n}), \alpha(\mathfrak{m})] \pm [g(\mathbf{z})\alpha(\mathfrak{n}), \alpha(\mathfrak{m})] = 0. \tag{3.8}$$

Right multiplying (8) by $\alpha(\mathfrak{m})$, we obtain

$$[T(\mathbf{z}\mathfrak{n})\alpha(\mathfrak{m}), \alpha(\mathfrak{m})] \pm [g(\mathbf{z})\alpha(\mathfrak{n})\alpha(\mathfrak{m}), \alpha(\mathfrak{m})] = 0. \tag{3.9}$$

Commuting (3.6) with $\alpha(\mathfrak{m})$, we have

$$[T(\mathbf{z}\mathfrak{n})\alpha(\mathfrak{m}) + \alpha(\mathbf{z}\mathfrak{n})d(\mathfrak{m}) \pm g(\mathfrak{m}\mathbf{z})\alpha(\mathfrak{n}), \alpha(\mathfrak{m})] = 0.$$

This yields

$$[T(\mathbf{z}\mathfrak{n})\alpha(\mathfrak{m}), \alpha(\mathfrak{m})] + [\alpha(\mathbf{z}\mathfrak{n})d(\mathfrak{m}), \alpha(\mathfrak{m})] \pm [g(\mathfrak{m}\mathbf{z})\alpha(\mathfrak{n}), \alpha(\mathfrak{m})] = 0. \tag{3.10}$$

Subtracting (3.9) from (3.10), we get

$$[\alpha(\mathbf{z}\mathfrak{n})d(\mathfrak{m}), \alpha(\mathfrak{m})] + [\pm g(\mathfrak{m}\mathbf{z})\alpha(\mathfrak{n}) \mp g(\mathbf{z})\alpha(\mathfrak{n})\alpha(\mathfrak{m}), \alpha(\mathfrak{m})] = 0. \tag{3.11}$$

Substituting $\mathfrak{n}\mathfrak{m}$ for $\mathfrak{n}$ in (3.11), we get

$$[\alpha(\mathbf{z}\mathfrak{n}\mathfrak{m})d(\mathfrak{m}), \alpha(\mathfrak{m})] + [\pm g(\mathfrak{m}\mathbf{z})\alpha(\mathfrak{n}) \mp g(\mathbf{z})\alpha(\mathfrak{n})\alpha(\mathfrak{m}), \alpha(\mathfrak{m})]\alpha(\mathfrak{m}) = 0. \tag{3.12}$$

Multiplying (3.11) by $\alpha(\mathfrak{m})$ on the right, we have

$$[\alpha(\mathbf{z}\mathfrak{n})d(\mathfrak{m}), \alpha(\mathfrak{m})]\alpha(\mathfrak{m}) + [\pm g(\mathfrak{m}\mathbf{z})\alpha(\mathfrak{n}) \mp g(\mathbf{z})\alpha(\mathfrak{n})\alpha(\mathfrak{m}), \alpha(\mathfrak{m})]\alpha(\mathfrak{m}) = 0. \tag{3.13}$$

Subtracting (3.12) from (3.13),

$$[\alpha(\mathbf{z}\mathfrak{n}\mathfrak{m})d(\mathfrak{m}), \alpha(\mathfrak{m})] - [\alpha(\mathbf{z}\mathfrak{n})d(\mathfrak{m}), \alpha(\mathfrak{m})]\alpha(\mathfrak{m}) = 0.$$

Since α is a homomorphism, we get

$$[\alpha(\mathbf{z}\mathfrak{n})\alpha(\mathfrak{m})d(\mathfrak{m}) - \alpha(\mathbf{z}\mathfrak{n})d(\mathfrak{m})\alpha(\mathfrak{m}), \alpha(\mathfrak{m})] = 0.$$

Since $\alpha(\mathbb{K}) = \mathbb{K}$,

$$[\mathbf{z}\mathfrak{n}[d(\mathfrak{m}), \alpha(\mathfrak{m})], \alpha(\mathfrak{m})] = 0. \tag{3.14}$$

Replacing $\mathbf{z}$ with $d(\mathfrak{m})\mathbf{z}$ in (3.14), we obtain

$$[d(\mathfrak{m})\mathbf{z}\mathfrak{n}[d(\mathfrak{m}), \alpha(\mathfrak{m})], \alpha(\mathfrak{m})] = 0.$$

This gives

$$[d(\mathfrak{m}), \alpha(\mathfrak{m})]\mathbf{z}\mathfrak{n}[d(\mathfrak{m}), \alpha(\mathfrak{m})] = 0. \tag{3.15}$$

Left multiplying (3.15) by $\mathbf{z}\mathfrak{n}$ implies

$$\mathbf{z}\mathfrak{n}[d(\mathfrak{m}), \alpha(\mathfrak{m})]\mathbb{R}\mathbf{z}\mathfrak{n}[d(\mathfrak{m}), \alpha(\mathfrak{m})] = 0. \tag{3.16}$$

The semiprimeness property of $\mathbb{R}$ yields

$$\mathbf{z}\mathfrak{n}[d(\mathfrak{m}), \alpha(\mathfrak{m})] = 0. \tag{3.17}$$

Since $\mathbb{K}$ is left ideal, we have $[d(\mathfrak{m}), \alpha(\mathfrak{m})]r\mathfrak{n} = 0$. In (3.17), replace $\mathbf{z}$ by $\mathfrak{n}$ and replace $\mathfrak{n}$ by $[d(\mathfrak{m}), \alpha(\mathfrak{m})]r\mathfrak{n}$, we find that

$$\mathfrak{n}[d(\mathfrak{m}), \alpha(\mathfrak{m})]r\mathfrak{n}[d(\mathfrak{m}), \alpha(\mathfrak{m})] = 0.$$

$$\mathbb{K}[d(\mathfrak{m}), \alpha(\mathfrak{m})] = 0.$$

References

[1] I.N. Herstein, Topics in ring theory, University of Chicago Press, 1969.

[2] M. Bresar, On the distance of the composition of two derivation to the generalized derivations, Glasgow Math. J., **33,** no. 1, (1991), 89–93.

[3] H.E. Bell, M. N. Daif, On centrally extended maps on rings, Beitrage Algebra Geom., Article no. 244, (2015), 1–8 .

[4] M.S. Tammam El-Sayiad, N.M. Muthana, Z.S. Alkhamisi, On rings with some Kinds of centrally-extended maps, Beitrage Algebra Geom., Article no. 274, (2015), 1–10.

[5] M.N. Daif, When is a multiplicative derivation additive, Int. J. Math. Math. Sci., **14,** no. 3, (1991), 615–618.

[6] M.N. Daif, M.S. Tammam El-Sayiad, Multiplicative generalized derivation which are additive, East-west J. Math., **19,** no. 1, (1997), 31–37.

[7] B. Dhara, S. Ali,on multiplicative (generalized) derivation, Aequat. Math., **86,** nos. 1-2, (2013), 65–79.

[8] A. K. Faraj, A. M. Abduldaim, Some Results Concerning Multiplicative (Generalized)-Derivations and Multiplicative Left Centralizers, Int. J. Math. Comput. Sci., **15,** no. 4, (2020), 1073–1090.

[9] M.N. Daif, M.S. Tammam El-Sayiad, V.D. Filippis, Multiplicative of left centralizers forcing additivity, Bol. Soc. Parana. Mat., **32,** no. 1, (2014), 61–69.

[10] I. N. Herstein, Jordan derivation of prime ring, Proc. Amer. Math. Soc., **8,** (1957), 1104–1110.

[11] A.K. Faraj, A.M. Abduldaim, Commutativity and Prime Ideals with Proposed Algebraic Identities, Int. J. Math. Comput. Sci., **16,** no. 4, (2021), 1607–1622.

[12] S.K. Tiwari, R.K. Sharma, B. Dhara, Some theorems of commutativity on semiprime ring with mapping, Southeast Asian Bull. Math., **42,** no. 2, (2018), 279–292.

[13] Z.S.M. Alhaidary, A.H. Majeed, Commutativity Results for Multiplicative (Generalizaed)(α,β) Reverse Derivations on Prime Rings, Iraqi Journal of Science, **62,** no. 9,(2021), 3102–3113.

[14] Z.S.M. Alhaidary, A.H. Majeed, Multiplicative (generalized) (η,ζ) reverse derivations on ideals of prime rings, AIP Conference Proceedings 2414, 040016; Published Online: February 13, 2023.

[15] Z.S.M. Alhaidary, A.H. Majeed, Square Closed Lie Ideals and Multiplicative (Generalized) (α,β) Reverse Derivation of Prime Rings, Journal of Discrete Mathematical Science and cryptography, (2021), 2037–2046.

[16] N. Muthana, Z. AlKhamisi, On centrally-extended multiplicative (generalized)-(α,β)-derivation in semiprime rings, Hacettepe Journal of Mathematics and Statistics, **49,** no. 2, (2021), 578–585.