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Differentiability of Mappings of Locally Convex Algebras

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Abstract

In this work, we introduce the notion of σ-differentiability and (A_1, σ)-differentiability of mappings in locally convex algebras. Moreover, we establish σ-continuous and (A_1, σ)-continuous mappings in these algebras associated with a system of bounded subsets in a convex algebra. In particular, we extend the method of Averbuch and Smolyanov in [1] to introduce a new approach of differentiability in topological algebras that is carried by a locally convex topology.

1 Introduction

In his doctoral thesis, Van Dantzig [2] introduced the theory of topological algebras in 1931. Since then, they were extensively studied by authors starting. A topological algebra is a topological vector space, separately or jointly

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equipped with continuous multiplication, making it also an algebra. Several papers and books have been written about the differential calculus in topological vector spaces (see [3], [6],[5], [4], [7] and [8]). To delve deeper into the specifics of differentiability in locally convex algebra, one can take a look at the sources that elaborate on the mathematical intricacies and properties of differentiable functions within these algebraic structures. The difficulty of constructing and defining differentiable functions in algebras is due to the topology in which the differentiable functions are continuous.
In this paper, we define the first difference, first variation, bounded element, and differentiability of locally convex algebra. Moreover, we give some properties of those concepts.

2 Preliminaries

A set $\mathcal{A}$ which is both a ring and a vector space over a field$K$ is called an algebra over K [7].

Let $(\Xi, +, \cdot)$ be a vector space over a field K. If Ξ is carried by a compatible topology τ (vector topology) in which the operations $+$ and $\cdot$ are continuous, then Ξ is called a topological vector space over K and is denoted by (Ξ, τ) [2]. A topological vector space which is also an algebra, such that the ring multiplication is separately continuous, is called a topological algebra[9]. A locally convex algebra is a topological algebra whose underlying topological vector space is a locally convex space; that is, with topology generated by system of locally convex of zero neighborhoods.

3 Differentiability and Continuity

Definition 3.1. *Let f be a map of a topological algebra $(\Xi_1, *, \tau_1)$ into a topological algebra $(\Xi_2, *, \tau_2)$. The first difference of f at a point $x \in \Xi_1$ for the increment $h \in \Xi_1$ is the expression $\Delta f(x, h) = f(x + h) - f(x)$*

Definition 3.2. *A map $f : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is called differentiable at the point $x \in \mathcal{A}_1$ in the direction $h \in \mathcal{A}_1$ if $\frac{\Delta f(x;\lambda h)}{\lambda}$ has a limit as $h \longrightarrow 0, \lambda \in R$. This limit is called the first variation (or simply the variation) of f at x for the increment h and is written as $\delta f(x, h)$. Therefore,*

$$\delta f(x, h) := \lim_{\lambda \longrightarrow 0} \frac{\Delta f(x; \lambda h) - f(x)}{\lambda} = \frac{\partial}{\partial x}|_{\lambda=0} f(x + \lambda h).$$

Clearly, if the variation for the increment h at x exists, then for any real number the variation for the increment αh at x exists and $\delta f(x;\alpha h) = \alpha\delta f(x;h)$.

Definition 3.3. *An element x of $\mathcal{A}_1$ is called bounded if, for some $0 \neq \alpha \in K$, the set $\{(\alpha x)^n : n = 1, 2, \ldots,\}$ is bounded. A subset A of $\mathcal{A}_1$ is bounded if each element of A is bounded.*
Now, let σ be a system of subsets of $\mathcal{A}_1$ and let β be a system of bounded subsets of $\mathcal{A}_2$. In addition, let $\pounds(\mathcal{A}_1, \mathcal{A}_2)$ be the vector space of all continuous linear mapping from $\mathcal{A}_1$ to $\mathcal{A}_2$ and let $\pounds_\beta(\mathcal{A}_1, \mathcal{A}_2)$ be a locally convex algebra with topology of uniformly convergence on a sets of β. Suppose that β contains of all singleton subsets of $\mathcal{A}_1$(singleton subsets are bounded) and that the union of sets in σ contains a neighborhood of zero in $\mathcal{A}_1$. From now on, a locally convex algebra is a subspace of algebra Ξ

Definition 3.4. *A map f from an algebra Ξ into a locally convex algebra $\mathcal{A}_2$ is said to be $\sigma-$ differentiable at $x \in \Xi$ along $\mathcal{A}_1$ (briefly $(\mathcal{A}_1, \sigma)-$differentiable) if there exist element $f^{'}(x) \in \pounds(\mathcal{A}_1, \mathcal{A}_2)$ such that for any $U \in \sigma$:*

$$\lim_{\lambda \longrightarrow 0} \frac{\Delta f(x;\lambda h)}{\lambda} = f^{'}(x).h$$

converges uniformly with respect to h in U. The map $f^{'} : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is called a $\sigma-$derivative at x along $\mathcal{A}_1$ (briefly $(\mathcal{A}_1, \sigma)-$derivative). The map $f^{'} : x \longrightarrow f^{'}(x)$ of Ξ in $\pounds_\beta(\mathcal{A}_1, \mathcal{A}_2)$ is called $\sigma\beta-$ derivative a along $\mathcal{A}_1$ (briefly $(\mathcal{A}_1, \sigma, \beta)-$derivative at x). Thus, if the map is $(\mathcal{A}_1, \sigma)-$differentiable at x, then the variation exists at this point for any increment h and $\delta f(x, h) = f^{'}(x)h$.

3.1 Example

Let f be a constant map from $\mathcal{A}_1$ into $\mathcal{A}_2$. Then f has a derivative $f^{'}(x) = 0$ in$\pounds(\mathcal{A}_1, \mathcal{A}_2)$ and f is $(\mathcal{A}_1, \sigma)-$differentiable at $x \in \mathcal{A}_1$.

3.2 Example

Any continuous linear map from $\mathcal{A}_1$ into $\mathcal{A}_2$ is $(\mathcal{A}_1, \sigma)-$derivative at x along $\mathcal{A}_1$ and $f^{'}(x) \equiv f$.

Remark 3.5. *For the map $\Xi \longrightarrow \mathcal{A}_2$ to be $(\mathcal{A}_1, \sigma)-$differentiable at x it is necessary and sufficient that,*

$$\frac{f(x + \lambda h) - f(x)}{\delta} \underset{h,\sigma}{\longrightarrow} \delta f(x;h)$$

The map $h \longrightarrow \delta f(x+\lambda h)$ is linear and $\delta f(x:\lambda h) \longrightarrow 0$ as $h \longrightarrow 0$. We observe that the definition of (H,σ)-differentiability at x can be written in the following equivalent form:

$$f(x+\lambda h) = f(x) + f^{'}(x)\lambda h + r(x,\lambda h) \tag{3.1}$$

where $f^{'}(x) \in \pounds(\mathcal{A}_1,\mathcal{A}_2)$ and $\frac{r(x,\lambda h)}{\lambda} \underset{h,\sigma}{\longrightarrow} 0$.

Proposition 3.6. *A map $\Xi \longrightarrow \mathcal{A}_2$ is boundedly $\mathcal{A}_1$-differentiable at x if and only if for any natural number n, $f(x+\lambda_1 h_1+\cdots+\lambda_n h_n) = f(x) + f^{'}(x)(\lambda_1 h_1+\cdots+\lambda_n h_n)+r(\lambda_1,\ldots,\lambda_n,h_1,\ldots,h_n)$ where for any bounded sets $B_1,\ldots,B_n$ in $\mathcal{A}_1$, the ratio $\frac{r(\lambda_1,\ldots,\lambda_n,h_1,\ldots,h_n)}{\|\lambda\|} \longrightarrow 0$ as $\parallel \lambda \parallel \longrightarrow 0$ uniformly in $h_i \in B_i, (i=1,\ldots,n)$, $\parallel \lambda \parallel^2 = \lambda_1^2+\cdots+\lambda_n^2$.*

Proof.

$$\begin{aligned} f(x+\lambda_1 h_1+\cdots+\lambda_n h_n) - f(x) &= f(x+\parallel \lambda \parallel (\frac{\lambda_1}{\parallel \lambda \parallel}h_1+\cdots+\frac{\lambda_n}{\parallel \lambda \parallel}h_n) - f(x) \\ &= f^{'}(x)(\lambda_1 h_1+\cdots+\lambda_n h_n)+ \\ & r(x,\parallel \lambda \parallel (\frac{\lambda_1}{\parallel \lambda \parallel}h_1+\cdots+\frac{\lambda_n}{\parallel \lambda \parallel}h_n). \end{aligned}$$

It is therefore enough to establish the boundedness of the set
$A = \{\frac{\lambda_1}{\|\lambda\|}h_1+\cdots+\frac{\lambda_n}{\|\lambda\|}h_n : \lambda_1,\ldots,\lambda_n \in R\}$.
Now $A \subseteq B_1+\cdots+B_n$ is bounded as the arithmetical sum of finite number of bounded sets. So the proposition follows. □

We now demonstrate a sufficient condition for the variation to be linear in the increment .

Proposition 3.7. *Let f be a map of Ξ into $\mathcal{A}_2$, $x \in \Xi$. For the map $h \longrightarrow \delta f(x:\lambda)$ of $\mathcal{A}_1$ into $\mathcal{A}_2$ to be linear it is sufficient that for any $h_1 \in \mathcal{A}_1$ the function of two real variables $\phi(\lambda_1,\lambda_2) = f(x+\lambda_1 h_1+\lambda_2 h_2)$ with values in $\mathcal{A}_2$ is boundedly $\mathcal{A}_1$-differentiable.*

Proof. It is enough to show that the map $h \longrightarrow \delta f(x;\lambda)$ is additive. On the one hand, by proposition 3.6, $f(x+\lambda_1 h_1+\lambda_2 h_2) = \phi(\lambda,\lambda) = \phi(0,0) = \phi(0,0)\lambda+\phi_2(0,0)\lambda+r(\lambda,\lambda) = f(x)+\delta f(x;h_1)\lambda+\delta f(x;h_2)\lambda+r(\lambda,\lambda)$, where $\frac{r(\lambda,\lambda)}{\sqrt{\lambda_1^2+\lambda_2^2}} \longrightarrow 0$ as $\lambda_1^2+\lambda_2^2 \longrightarrow 0$. on the other hand, by equation 3.1, we have $\delta f(x;h_1+h_2) = \delta f(x;h_1)+\delta f(x;h_2)$ where $\frac{r(\lambda)}{\lambda} \longrightarrow 0$ as $\lambda \longrightarrow 0$. Comparing, we obtain $\delta f(x;h_1+h_2) = \delta f(x;h_1)+\delta f(x;h_2)$. □

Definition 3.8. *A map* $f : \Xi \longrightarrow \mathcal{A}_2$ *is continuous at* $x \in \Xi$ *along* $\mathcal{A}_1$ *(briefly* $\mathcal{A}_1$*-continuous) if the map* $h \longrightarrow f(x+h)$ *of* $\mathcal{A}_1$ *into* $\mathcal{A}_2$ *is continuous at zero.*

Definition 3.9. *A map* $f : \Xi \longrightarrow \mathcal{A}_2$ *is* σ*- continuous at* $x \in \Xi$ *along* $\mathcal{A}_1$*, (briefly* $\mathcal{A}_1$*-continuous at* x *if* $f(x + \lambda h) \underset{h,\sigma}{\longrightarrow} f(x)$*. We simply write* σ*-continuous instead of* $(\mathcal{A}_1, \sigma)$*- continuous if* $\mathcal{A}_1 = \Xi$ *.*

Proposition 3.10. *Let a map* $f : \Xi \longrightarrow \mathcal{A}_2$ *be* $(\mathcal{A}_1, \beta)-$*differentiable at* x*, where* β *is a system of bounded subsets of* $\mathcal{A}_1$*. Then it is* $(\mathcal{A}_1, \beta)-$*continuous at this point.*

Proof. The equation $f(x+\lambda h)-f(x) = \lambda f^{'}(x)h+\lambda r(\lambda, h)$ with $r(\lambda, h), \beta \longrightarrow 0$ follows from the definition of $(\mathcal{A}_1, \beta)-$differentiability. Let $B \in \beta$ the set $\{f^{'}(x)h : h \in \beta\}$ is bounded, as the image of a bounded set under a linear continuous map. Consequently, $f^{'}(x)h \longrightarrow 0$ as $\lambda \longrightarrow 0$ uniformly for $h \in \beta$. Furthermore, $\lambda r(\lambda, h) \longrightarrow 0$, as $\lambda \longrightarrow 0$ uniformly for $h \in \beta$ □

Proposition 3.11. *If the map* $f : \Xi \longrightarrow \mathcal{A}_2$ *is* $\mathcal{A}_1$*-continuous at* x *, then it is* $(\mathcal{A}_1, \beta)-$ *continuous at this point for any system* β *of bounded subsets of* $\mathcal{A}_1$*.*

Proof. Let the map f be $\mathcal{A}_1$-continuous. Then for any neighborhood N of zero in $\mathcal{A}_2$ there is a neighborhood V of zero in $\mathcal{A}_1$ such that $f(x+h^{'})-f(x) \in N$ if $h^{'} \in \mathcal{A}_2$. Then if B is a bounded subsets of $\mathcal{A}_1$, there is a $\lambda_u > 0$ such that $\lambda B \in \mathcal{A}_2$ for $|\lambda| < \lambda_u$ such that $f(x + \lambda h) - f(x) \in N$ and $h \in B$.

□

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