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A Note on the Exponential Diophantine Equation $8^x + 161^y = z^2$

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Abstract

In this note, we revisit the exponential Diophantine equation $8^x + 161^y = z^2$, initially studied by Manikandan and Venkatraman. Their work established that the equation has the two non-negative integer solutions: $(1, 0, 3)$ and $(1, 1, 13)$. Our findings reveal an additional solution, $(2, 1, 15)$, and we show that these three solutions constitute the complete list of non-negative integer solutions for this equation. This extends and completes the main result presented in their paper.

1 Introduction

Exponential Diophantine equations of the form $a^x + b^y = z^2$ have garnered considerable interest due to their rich mathematical properties and complexity. In their recent publication, Manikandan and Venkatraman [1] analyzed the equation $8^x + 161^y = z^2$ and concluded that $(1, 0, 3)$ and $(1, 1, 13)$ are non-negative integer solutions. Our findings reveal an additional solution, $(2, 1, 15)$, thereby extending the main result presented in their paper.

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2 Main results

The following lemma has been proved in Lemma 2.3 in Manikandan and Venkatraman's article [1] using Mihăilescu's theorem. Here, we provide a proof using only elementary methods.

Lemma 2.1. *The Diophantine equation* $1 + 161^y = z^2$ *has no non-negative integer solutions.*

Proof. Assume $1 + 161^y = z^2$ for non-negative integers y and z. If $y = 0$, then $z^2 = 2$, which is impossible. If $z = 0$, then $161^y = -1$, which is also impossible. For $y > 0$ and $z > 0$, we have $z^2 - 1 = 161^y$.

Factoring, we get $(z - 1)(z + 1) = 161^y$. We have $z \geq 8$. Since $z - 1$ and $z + 1$ are consecutive odd numbers, they must be coprime. Given that $161 = 7 \cdot 23$ and $z - 1 < z + 1$, it follows that $z - 1 = 7^y$ and $z + 1 = 23^y$. However, 7^y and 23^y cannot be consecutive odd integers for any positive integer y. Therefore, the equation $1 + 161^y = z^2$ has no non-negative integer solutions. □

We now state the following lemma:

Lemma 2.2. *[3] The Diophantine equation* $8^x + 1 = z^2$ *has no non-negative integer solutions other than* $(1, 3)$.

The following theorem from Scott and Styer's work [4] provides a key insight into our problem.

Theorem 2.3. *[4] For relatively prime integers* a *and* b *both greater than one and odd integer* c*, there are at most two solutions in positive integers* (x, y, z) *to the equation* $a^x + b^y = c^z$*. Any solution* (x, y, z) *must satisfy* $z < \dfrac{ab}{2}$.

Using this theorem as a basis, we present our main result regarding the equation $8^x + 161^y = z^2$.

2.1 Main Theorem

Theorem 2.4. *The exponential Diophantine equation* $8^x + 161^y = z^2$ *has exactly the three non-negative integer solutions:* $(1, 0, 3)$, $(1, 1, 13)$, *and* $(2, 1, 15)$.

Proof. Clearly, $(1, 0, 3)$, $(1, 1, 13)$, and $(2, 1, 15)$ are solutions.

We will now show that there are no other solutions. According to Lemma 2.1, if $x = 0$, then $8^0 + 161^y = z^2$ implies $1 + 161^y = z^2$, which has no non-negative integer solutions. Hence, the only valid case is $y = 0$ and $z = 1$, leading to $(1, 0, 3)$.

If $y = 0$, then $8^x + 1 = z^2$. Lemma 2.2 shows that the only solution for $x \geq 1$ is $(1, 0, 3)$.

For cases where both x and y are positive, Theorem 2.3 confirms that for relatively prime integers a and b and an odd integer c, there are at most two solutions in positive integers (x, y, z). This is consistent with our findings.

Therefore, the equation $8^x + 161^y = z^2$ has exactly three non-negative integer solutions. $\square$

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