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Euler-type formulas

Badih Ghusayni

Department of Mathematics
Faculty of Science-1
Lebanese University
Hadath, Lebanon

email: badih@future-in-tech.net

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Abstract

Finding the exact value that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges to is one of the most notorious problems that did not even yield to Euler. A less difficult problem to consider is to find a representation of $\zeta(3)$ in terms of

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}$$

only which would be a beautiful result similar to Euler's

$$\zeta(2) = 3\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}}$$

not to mention the fact that it is faster converging.
In this paper we find a new representation of $\zeta(3)$ in terms of $\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}$ and $\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{x} dx$. More precisely, we prove the following new formula:

$$\zeta(3) = \pi \int_0^{\frac{1}{2}} \frac{\sin^{-1} y}{y} dy - \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}.$$

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1 Introduction

The formula

$$\zeta(3) = \frac{2}{7}\pi^2 \log 2 + \frac{16}{7}\int_0^{\frac{\pi}{2}} x \log(\sin x) dx \qquad (1)$$

was discovered by Euler [5] who by the time he proved this in 1772 had been blind for 6 years according to [1], p. 1084.
From ([7], formula 3):

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}} = -2\int_0^{\frac{\pi}{3}} x \log(2\sin(\frac{x}{2}))dx$$

which is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}} = -8\int_0^{\frac{\pi}{6}} u \log(2\sin u)du = -\frac{1}{9}\pi^2 \log 2 - 8\int_0^{\frac{\pi}{6}} u \log(\sin u)du$$

suggests a possible connection between $\zeta(3)$ and $\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}$ only.

A variant of formula (1) is ([4], formula 89)

$$\zeta(3) = \frac{2}{9}\pi^2 \log 2 + \frac{16}{3\pi}\int_0^{\frac{\pi}{2}} x^2 \log(\sin x)dx$$

Euler [6] showed that

$$\int_0^{\frac{\pi}{2}} \log(\sin x)dx = -\frac{1}{2}\pi \log 2.$$

Using integration by parts we have

$$\int_0^t x^2 \cot x\, dx = x^2 \log(\sin x)\Big|_0^t - 2\int_0^t x \log(\sin x)\, dx$$

Since $x^2 \log(\sin x) = x^2 \log \frac{\sin x}{x} + x^2 \log x$ we have $\lim_{x\to 0} x^2 \log \sin x = 0$. As a result,

$$\int_0^t x^2 \cot x\, dx = t^2 \log(\sin t) - 2\int_0^t x \log(\sin x)\, dx$$

which for $t = \frac{\pi}{2}$ yields

$$\int_0^{\frac{\pi}{2}} x^2 \cot x \, dx = -2 \int_0^{\frac{\pi}{2}} x \log(\sin x) \, dx$$

and consequently we get

$$\zeta(3) = \frac{2}{7}\pi^2 \log 2 - \frac{8}{7} \int_0^{\frac{\pi}{2}} x^2 \cot x dx,$$

a formula where the integrand is free of log.
Again, a variant of this formula ([4], formula 42) is

$$\zeta(3) = \frac{2}{9}\pi^2 \log 2 - \frac{16}{9\pi} \int_0^{\frac{\pi}{2}} x^3 \cot x dx.$$

It is worth mentioning here the well-known Euler integral ([4], p. 54)

$$\int_0^{\frac{\pi}{2}} x \cot x dx = \frac{1}{2}\pi \log 2 \tag{2}$$

and ([9], p. 82)

$$\int_0^{\frac{\pi}{2}} \log(\cos x) dx = \int_0^{\frac{\pi}{2}} \log(\sin x) dx = -\frac{1}{2}\pi \log 2 \tag{3}$$

On the other hand ([4], formula 102,)

$$\int_0^{t} \frac{x}{\sin x} dx = t \log \tan\left(\frac{t}{2}\right) + 2 \sum_{n=0}^{\infty} \frac{\sin(2n+1)t}{(2n+1)^2} \tag{4}$$

Again, it is worth mentioning here that

$$\int_0^{\frac{\pi}{2}} \frac{x^2}{\sin^2 x} dx = \pi \log 2.$$

However, $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{x}{\sin x} dx = G$, Catalan's constant which is still unknown.
Together with [2], formula (35)

$$\int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x} dx = 2\pi G - \frac{7}{2}\zeta(3)$$

yield the curiously looking formula

$$\zeta(3) = \frac{2}{7} \int_0^{\frac{\pi}{2}} \frac{x(\pi - x)}{\sin x} dx.$$

2 The Main problem

In [7] I have obtained the following representation

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{3}}{n^2} - \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^3 \begin{pmatrix} 2n \\ n \end{pmatrix}}$$

which in [2] I have improved to

$$\zeta(3) = -\frac{\sqrt{3}}{18}\pi^3 + \frac{3\sqrt{3}}{4}\pi \sum_{n=1}^{\infty} \frac{1}{(3n-2)^2} - \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^3 \begin{pmatrix} 2n \\ n \end{pmatrix}} \quad (5)$$

The simply-looking series $\sum_{n=1}^{\infty} \frac{1}{(3n-2)^2}$, however, is hard to evaluate as I came to realize. Indeed, it appeared often even in Ramanujan's writings (see, for example, [3]) and specifically as

$$\int_0^{\frac{1}{\sqrt{3}}} \frac{\tan^{-1} t}{t} dt = -\frac{\pi}{12} \log 3 - \frac{5\pi^2}{18\sqrt{3}} + \frac{5\sqrt{3}}{4} \sum_{k=0}^{\infty} \frac{1}{(3k+1)^2} \quad (6)$$

The alternative venue in lieu of the value that $\sum_{n=1}^{\infty} \frac{1}{(3n-2)^2}$ converges to is to express $\sum_{n=1}^{\infty} \frac{1}{(3n-2)^2}$ in terms of either $\zeta(3)$ or $\sum_{n=1}^{\infty} \frac{1}{n^3 \begin{pmatrix} 2n \\ n \end{pmatrix}}$ thus attaining the objective of expressing $\zeta(3)$ in terms of $\sum_{n=1}^{\infty} \frac{1}{n^3 \begin{pmatrix} 2n \\ n \end{pmatrix}}$ only. However, I was not able to do that mathematically or even by using the Computer Algebra System Maple. This gridlock motivated this paper to obtain a formula similar to formula (5) with the term $\sum_{n=1}^{\infty} \frac{1}{(3n-2)^2}$ replaced with an easier one.

Looking at formula (6) the idea is to use the LLL algorithm [10] incorporated in Maple to try to find an integer relation among say $\zeta(3)$, $\int_0^{\frac{1}{\sqrt{3}}} \frac{\tan^{-1} x}{x} dx$ and $\sum_{n=1}^{\infty} \frac{1}{n^3 \begin{pmatrix} 2n \\ n \end{pmatrix}}$ which can then possibly proven mathematically.

The previous discussion suggested the following Maple worksheet:

> a := evalf((Pi*sqrt(3))*(sum(1/(3*n+1)∧2, n = 0 .. infinity)))

$a := 6.103795887$

> b := evalf(Zeta(3))
b := 1.202056903
> c := evalf(sum(1/(n∧3*binomial(2*n, n)), n = 1 .. infinity))
c := 0.5229461921
f := evalf(Pi*(Int(arctan(t)/t, t = 0 .. 1/sqrt(3))))
f := 1.753538522
> A := trunc(10∧10*a)
A := 61037958870
> B := trunc(10∧10*b)
B := 12020569030
> C := trunc(10∧10*c); F := trunc(10∧10*f)
C := 5229461921
F := 17535385220
> v1 := [A, 1, 0, 0, 0]
$v1 := [61037958870, 1, 0, 0, 0]$
> v2 := [B, 0, 1, 0, 0]
$v2 := [12020569030, 0, 1, 0, 0]$ > v3 := [C, 0, 0, 1, 0]; v4 := [F, 0, 0, 0, 1]
$v3 := [5229461921, 0, 0, 1, 0]$
$v4 := [17535385220, 0, 0, 0, 1]$
> with(IntegerRelations)
$[LLL, LinearDependency, PSLQ]$
> LLL([v1, v2, v3, v4]); [[-12, 12, 63, -362, 23], [50, 84, -453, -80, 42], [264, -103, 30, 134, 298],
[-763, -89, -251, 137, 441]]

Unfortunately, the presence of the "relatively large" numbers, say $12, 63, -362, 23$ did not look promising.
Luckily, when I changed $\tan^{-1} x$ to $\sin^{-1} x$ with x in the right interval $(0, \frac{1}{2})$ (clearly, $\sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1-x^2}}$ for $x^2 < 1$), the gloomy picture turned around as the following worksheet shows (incidentally, I added more digits for a change-but that is not required-and I checked the result with Maple after small numbers came out):

> Digits := 60
Digits:=60
>a := evalf(Pi*sqrt(3)*sum(1/(3*n+1)∧2, n = 0 .. infinity)
a := 6.10379588255482017770126335812878999604878415239410822371063
> b := evalf(Zeta(3))
b := 1.20205690315959428539973816151144999076498629234049888179227

> c := evalf(sum(1/(n∧3*binomial(2*n, n)), n = 1 ..infinity))
c := 0.52294619213333510849118518352730354016304459174397784146594 1
> f := evalf(Pi*(Int(arcsin(t)/t, t = 0..1/2)));
f := 1.5942665472595956167681270491569276458872697361484822628917 3
> A := trunc(10∧10*a)
A := 61037958825
> B := trunc(10∧10*b)
B := 12020569031
> C := trunc(10∧10*c); F := trunc(10∧10*f)
C := 5229461921
F := 15942665472
> v1 := [A, 1, 0, 0, 0]
$v1 := [61037958825, 1, 0, 0, 0]$
> v2 := [B, 0, 1, 0, 0]
$v2 := [12020569031, 0, 1, 0, 0]$
> v3 := [C, 0, 0, 1, 0]; v4 := [F, 0, 0, 0, 1]
$v3 := [5229461921, 0, 0, 1, 0]$
$v4 := [15942665472, 0, 0, 0, 1]$
> with(IntegerRelations)
[LLL, LinearDependency, PSLQ]
> LLL([v1, v2, v3, v4])
$[[1, 0, -4, -3, 4], [1178, 428, -1045, 449, -998],$

$[753, -49, 1373, -1889, -228], [938, -480, 55, 1833, 1195]]$

Thus the formula we obtained with Maple as a tool is:

$$-4\zeta(3) - 3\sum_{n=1}^{\infty} \frac{1}{n^3 \begin{pmatrix} 2n \\ n \end{pmatrix}} + 4\pi \int_0^{\frac{1}{2}} \frac{\sin^{-1} t}{t} dt = 0.$$

To test the result obtained using Maple we have:
> evalf(-4*Zeta(3)-3*(sum(1/(n∧3binomial(2*n,n)),n=1..infinity))+ 4*Pi*(Int(arcsin(t)/t, t=0..1/2)))

2.10^{-59}

Note that the last step is reassuring.

We now supply a mathematical proof of the discovered identity:

$$\zeta(3) = \pi \int_0^{\frac{1}{2}} \frac{\sin^{-1} y}{y} dy - \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}$$

By the main result in [7], it is enough to show that

$$\int_0^{\frac{1}{2}} \frac{\sin^{-1} y}{y} dy = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{3}}{n^2}.$$

Clearly,

$$\int u \cot u du \overbrace{=}^{w=\sin u} \int \frac{\sin^{-1} w}{w} dw.$$

In particular,

$$\int_0^{\sin t} \frac{\sin^{-1} y}{y} dy = \int_0^t x \cot x dx$$

(Using formula (2) we get the curious special case $\int_0^1 \frac{\sin^{-1} y}{y} dy = \frac{1}{2}\pi \log 2$.) Therefore, using a formula in ([4], p. 53)

$$\int_0^{\frac{1}{2}} \frac{\sin^{-1} y}{y} dy = \int_0^{\frac{\pi}{6}} x \cot x dx = 2 \sum_{n=0}^{\infty} \int_0^{\frac{\pi}{6}} x \cos x \sin(2n+1) x dx$$

$$= \int_0^{\frac{\pi}{6}} x \sin(2x) dx + 2 \sum_{n=1}^{\infty} \int_0^{\frac{\pi}{6}} x \cos x \sin(2n+1) x dx$$

where the first term evaluates to $-\frac{\pi}{24} + \frac{\sqrt{3}}{8}$ and the second term yields, upon integration by parts,

$$\sum_{n=1}^{\infty} \left(\frac{\sin \frac{n\pi}{3}}{4n^2} - \frac{\pi}{12} \frac{\cos \frac{n\pi}{3}}{n} + \frac{\sin \frac{(n+1)\pi}{3}}{4(n+1)^2} - \frac{\pi}{12} \frac{\cos \frac{(n+1)\pi}{3}}{n+1} \right)$$

Now by ([7], p. 174) for $x \in (0, 2\pi)$,

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = -\log(2 \sin \frac{x}{2})$$

and so $\sum_{n=1}^{\infty} \frac{\cos \frac{n\pi}{3}}{n} = 0$ and $\sum_{n=1}^{\infty} \frac{\cos \frac{(n+1)\pi}{3}}{n+1} = -\cos \frac{\pi}{3} = -\frac{1}{2}$. In addition,

$$\sum_{n=1}^{\infty} \frac{\sin \frac{(n+1)\pi}{3}}{4(n+1)^2} = \sum_{n=2}^{\infty} \frac{\sin \frac{n\pi}{3}}{4n^2} = \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{3}}{4n^2} - \frac{\sqrt{3}}{8}.$$

Consequently,

$$\int_0^{\frac{1}{2}} \frac{\sin^{-1} y}{y} dy = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{3}}{n^2}.$$

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