Absolute Summability Factors Involving Quasi-$f$-Power Increasing Sequences

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Abstract

Two new general results concerning absolute summability of an infinite series which extended and generalized the result of Sevli and Leindler [4] are achieved.

1 Introduction

Let $T$ be a lower triangular matrix, $(s_n)$ a sequence of the $n$th partial sums of the series $\sum a_n$, and

$$T_n := \sum_{v=0}^{n} t_{nv}s_v.$$ (1.1)

A series $\sum a_n$ is said to be summable $|T|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta T_{n-1}|^k < \infty.$$ (1.2)

Given any lower triangular matrix $T$, one can associate the matrices $\bar{T}$ and $\hat{T}$, with entries defined by $\bar{t}_{nv} = \sum_{i=1}^{n} t_{ni}$, $n, i = 0, 1, 2, \ldots$, $\hat{t}_{nv} =$

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\( \tilde{t}_{nv} - \tilde{t}_{n-1,v} \) respectively. With \( s_n = \sum_{i=0}^{n} a_i \lambda_i \),

\[
t_n = \sum_{v=0}^{n} t_{nv} s_v = \sum_{v=0}^{n} t_{nv} \sum_{i=0}^{v} a_i \lambda_i = \sum_{i=0}^{n} a_i \lambda_i \sum_{v=0}^{n} t_{nv} = \sum_{i=0}^{n} \tilde{t}_{ni} a_i \lambda_i. \tag{1.3}
\]

\[
Y_n := t_n - t_{n-1} = \sum_{i=0}^{n} \tilde{t}_{ni} a_i \lambda_i - \sum_{i=0}^{n-1} \tilde{t}_{n-1,i} a_i \lambda_i = \sum_{i=0}^{n} \tilde{t}_{ni} a_i \lambda_i, \quad \text{as } \tilde{t}_{n-1,n} = 0. \tag{1.4}
\]

Let \( \Omega \) be the class of all matrices \( T = (t_{nv}) \) satisfying

(i) \( T \) is lower triangular,

(ii) \( t_{nv} \geq 0, \; n, v = 0, 1, \ldots \),

(iii) \( t_{n-1,v} \geq t_{nv}, \; n \geq v + 1 \),

(iv) \( \tilde{t}_{n0} = 1, \; n = 0, 1, \ldots \).

A positive sequence \( \gamma = (\gamma_n) \) is said to be a quasi-\( \beta \)-power increasing sequence if there exists a constant \( K = K(\beta, \gamma) \geq 1 \) such that

\[
Kn^\beta \gamma_n \geq m^\beta \gamma_m, \tag{1.5}
\]

holds for \( n \geq m \geq 1 \).

A positive sequence \( \gamma = (\gamma_n) \) is said to be a quasi-\( f \)-power increasing sequence, where \( f = (f_n) = (n^\beta (\log n)^\mu), \; 0 \leq \beta < 1, \; \mu \geq 0 \) if (see[5]) there exists a constant \( K = K(\gamma, f) \geq 1 \) such that

\[
K f_n \gamma_n \geq f_m \gamma_m, \tag{1.6}
\]

holds for \( n \geq m \geq 1 \).

Let \( t_n \) denote the \( n \)th (C,1) mean of the sequence \( (na_n) \), that is

\[
t_n = \frac{1}{n+1} \sum_{v=1}^{n} va_v. \tag{1.7}
\]

Very recently Sevli and Leindler [4] proved the following result

**Theorem 1.1.** \( T \in \Omega \) satisfying

\[
nt_{mn} = O(1), \quad n \to \infty, \tag{1.8}
\]
and let \((\lambda_n)\) be a sequence of real numbers satisfying

\[ m \sum_{n=1}^{m} \lambda_n = o(m), \quad m \to \infty, \quad (1.9) \]

\[ m \sum_{n=1}^{m} |\Delta \lambda_n| = o(m), \quad m \to \infty. \quad (1.10) \]

If \((X_n)\) is a quasi-\(f\)-power increasing sequence such that the conditions

\[ \sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m), \quad m \to \infty, \quad (1.11) \]

\[ \sum_{n=1}^{\infty} n X_n(\beta, \mu) |\Delta |\Delta \lambda_n|| < \infty, \quad (1.12) \]

are satisfied, then the series \(\sum a_n \lambda_n\) is summable \(|T|_k\), \(k \geq 1\), where \((f_n) := (n^{3/2} (\log n)^\mu)\), \(\mu \geq 0\), \(0 \leq \beta < 1\), and \(X_n(\beta, \mu) = \max(n^{3/2} (\log n)^\mu, \log n)\).

## 2 Lemmas

**Lemma 2.1** [1]. Let \(T \in \Omega\), then

\[ \hat{t}_{n,v+1} \leq t_{nn}, \quad n \geq v + 1, \quad (2.13) \]

and

\[ \sum_{n=v+1}^{m+1} \hat{t}_{n,v+1} \leq 1, \quad n = 0, 1, ... \quad (2.14) \]

**Lemma 2.2** [1]. Let \(T \in \Omega\), then

\[ \sum_{v=1}^{n-1} |\Delta_v \hat{t}_{nv}| \leq t_{nn}, \quad (2.15) \]
and
\[ \sum_{n=v+1}^{m+1} |\Delta \tilde{t}_n| \leq t_{vv}. \] (2.16)

**Lemma 2.3.** If \((\lambda_n)\) is convex sequence such that \(\sum n^{-1} \lambda_n\) is convergent, and the following condition
\[ \sum_{n=1}^{\infty} n^\beta (\log n)^\gamma X_n |\Delta |\Delta \lambda_n|| < \infty, \] (2.17)
where
\[ \beta = \begin{cases} 1, & \lambda > 1, \\ > 1, & 0 \leq \gamma \leq 1, \end{cases} \] (2.18)
holds, then the following inequalities hold
\[ nX_n |\Delta \lambda_n| = O(1), \quad n \to \infty, \] (2.19)
\[ \sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty. \] (2.20)

**Proof.** Since \(\Delta \lambda_n \to 0\) as \(n \to \infty\), and \((n^\beta (\log n)^\gamma X_n)\) is non-decreasing,
\[ \sum_{n=1}^{\infty} X_n |\Delta \lambda_n| = \sum_{n=1}^{\infty} X_n \sum_{v=1}^{\infty} \Delta |\Delta \lambda_v| \leq \sum_{n=1}^{\infty} X_n \sum_{v=1}^{\infty} |\Delta |\Delta \lambda_v|| = \]
\[ \sum_{v=1}^{\infty} |\Delta |\Delta \lambda_v|| \sum_{n=1}^{\infty} X_n \sum_{v=1}^{\infty} \Delta |\Delta \lambda_v|| \sum_{n=1}^{\infty} n^{-\beta (\log n)^\gamma} = O(1) \sum_{v=1}^{\infty} v^\beta (\log v)^\gamma X_v |\Delta |\Delta \lambda_v|| \sum_{n=1}^{\infty} n^{-\beta (\log n)^\gamma} = \]
\[ O(1) \sum_{v=1}^{\infty} v^\beta (\log v)^\gamma X_v |\Delta |\Delta \lambda_v|| \sum_{n=1}^{\infty} n^{-\beta (\log n)^\gamma} = O(1) \sum_{v=1}^{\infty} v^\beta (\log v)^\gamma X_v |\Delta |\Delta \lambda_v|| = \]
\[ O(1). \]

3 Main result

The object of this paper is to give two general results in both we are extending the range of \(\beta\). Theorem 1.1 is dealing with all \(\beta\) such that \(0 \leq \beta < 1\). Our results are dealing with all \(\beta\) such that \(\beta \geq 1\).
Theorem 3.1. Let $T \in \Omega$ satisfy (1.8) and let $(\lambda_n)$ be a sequence of bounded variation such that $\sum n^{-1} \lambda_n$ is convergent. Let (1.11) and (2.17) be satisfied and

$$|\lambda_n| X_n = O(1), \quad \text{as } n \to \infty,$$

(3.21)

where $(X_n)$ is a quasi-$f$-power increasing sequence, $f = (f_n) = (n^{\beta}(\log n)^{\gamma})$, where $\beta$, $\gamma$ satisfying (2.18), then the series $\sum a_n \lambda_n$ is summable $|T|_k$, $k \geq 1$.

Proof. By Abel’s transformation, we have

$$Y_n = \sum_{v=1}^{n} \hat{t}_{nv} \lambda_v = \sum_{v=1}^{n} v a_v v^{-1} t_{nv} \lambda_v$$

$$= \sum_{v=1}^{n-1} \left( \sum_{r=1}^{n} ra_r \right) \Delta_v (v^{-1} \hat{t}_{nv} \lambda_v) + n^{-1} t_{nn} \lambda_n \sum_{v=1}^{n} va_v$$

$$= \sum_{v=1}^{n-1} (v+1)t_v \left( \frac{1}{v(v+1)} \hat{t}_{nv} \lambda_v + \frac{1}{(v+1)} \Delta_v \hat{t}_{nv} \lambda_v + \frac{1}{(v+1)} \hat{t}_{n,v+1} \Delta \lambda_v \right)$$

$$+ \frac{n+1}{n} t_{nn} \lambda_n t_n$$

$$= \sum_{v=1}^{n-1} v^{-1} \hat{t}_{nv} \lambda_v t_v + \sum_{v=1}^{n-1} \Delta_v \hat{t}_{nv} \lambda_v t_v + \sum_{v=1}^{n-1} t_v \hat{t}_{n,v+1} \Delta \lambda_v + \frac{n+1}{n} t_{nn} \lambda_n t_n$$

$$= Y_{n_1} + Y_{n_2} + Y_{n_3} + Y_{n_4}.$$

In order to complete the proof of the theorem, by Minkowski’s inequality, it is therefore sufficient to show that $\sum_{n=1}^{m} a_n |Y_{nj}|^k < \infty$ $j = 1, 2, 3, 4$. Applying Hölder’s inequality, we have, via (3.21), (1.8), (2.14) and (2.20),

$$\sum_{n=1}^{m+1} n^{k-1} |Y_{n1}|^k = \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{v=1}^{n-1} v^{-1} \hat{t}_{nv} \lambda_v t_v \right|^k$$

$$\leq \sum_{n=1}^{m+1} n^{k-1} \sum_{v=1}^{n-1} v^{-1} |\hat{t}_{nv}|^k |\lambda_v| |t_v|^k \left( \sum_{v=1}^{n-1} v^{-1} |\lambda_v| \right)^{k-1}$$
\[= O(1) \sum_{n=1}^{m} n^{k-1} |\hat{\lambda}_v| \sum_{v=1}^{k} n^{-1} |\lambda_v||t_v|^k\]

\[= O(1) \sum_{v=1}^{m} v^{-1} |\lambda_v||t_v|^k \sum_{n=v+1}^{m+1} n^{k-1} |t_{nn}|^{-k-1} |\hat{\lambda}_{nnv}|\]

\[= O(1) \sum_{v=1}^{m} v^{-1} |\lambda_v||t_v|^k = O(1) \sum_{v=1}^{m} v^{-1} |\lambda_v||t_v|^k\]

\[= O(1) \sum_{v=1}^{m} \left( \sum_{r=1}^{v} r^{-1} |t_v|^k \right) \Delta |\lambda_v| + O(1) |\lambda_m| \sum_{v=1}^{m} v^{-1} |t_v|^k\]

\[= O(1) \sum_{v=1}^{m} X_v |\Delta \lambda_v| + O(1) |\lambda_m| X_m = O(1).\]

By (2.15), (1.8) and (2.16), \[\sum_{n=1}^{m+1} n^{-1} |Y_{n2}|^k = \sum_{n=1}^{m+1} n^{k-1} |\sum_{v=1}^{n-1} \Delta_v \hat{\lambda}_{nv} \lambda_v t_v|^k \leq \sum_{n=1}^{m+1} n^{k-1} \sum_{v=1}^{n-1} |\Delta_v \hat{\lambda}_{nv}| |\lambda_v|^k |t_v|^k (\sum_{v=1}^{n-1} |\Delta_v \hat{\lambda}_{nv}|)^{k-1} = O(1) \sum_{n=1}^{m+1} n^{k-1} (t_{nn})^{k-1} \sum_{v=1}^{n-1} |\Delta_v \hat{\lambda}_{nv}| |\lambda_v|^k |t_v|^k.\]

By (1.8) and (2.14),

\[\sum_{n=1}^{m+1} n^{k-1} |Y_{n3}|^k = \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{v=1}^{n-1} t_v \hat{\lambda}_{n,v+1} \Delta \lambda_v \right|^k \]

\[\leq \sum_{n=1}^{m+1} n^{k-1} \sum_{v=1}^{n-1} |t_v|^k |\hat{\lambda}_{n,v+1}|^k |\Delta \lambda_v|^k \left( \sum_{v=1}^{n-1} |\Delta \lambda_v|^k \right)^{k-1} = O(1) \sum_{v=1}^{m} |t_v|^k |\Delta \lambda_v| \sum_{n=v+1}^{m+1} n^{k-1} |t_{n,v+1}|^k\]
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\[
= O(1) \sum_{v=1}^{m} |t_v|^k |\Delta \lambda_v| \sum_{n=v+1}^{m+1} (nt_{mn})^{k-1} |\hat{t}_{n,v+1}|
\]

\[
= O(1) \sum_{v=1}^{m} |t_v|^k |\Delta \lambda_v| \sum_{n=v+1}^{m+1} |\hat{t}_{n,v+1}|
\]

\[
= O(1) \sum_{v=1}^{m} |t_v|^k |\Delta \lambda_v| = O(1) \sum_{v=1}^{m} v^{-1} |t_v|^k v |\Delta \lambda_v|
\]

\[
= O(1) \sum_{v=1}^{m} X_v |\Delta \lambda_v| + O(1) \sum_{v=1}^{m} v X_v |\Delta \lambda_v| + O(1) m X_m |\Delta \lambda_m| = O(1).
\]

Finally, by (1.8),

\[
\sum_{n=1}^{m} n^{-1} |Y_{n4}|^k = \sum_{n=1}^{m} n^{k-1} \left| \frac{n+1}{n} t_{nn} \lambda_n t_n \right|^k
\]

\[
= O(1) \sum_{n=1}^{m} (nt_{nn})^{k-1} t_{nn} |\lambda_n|^k |t_n|^k = O(1) \sum_{n=1}^{m} n^{-1} |\lambda_n| |t_n|^k = O(1), \text{ as in the case of } Y_{n1}.
\]

**Theorem 3.2.** Let \( T \in \Omega \) satisfy (1.8) and let \((\lambda_n)\) be a sequence of real numbers satisfying \( \Delta \lambda_n \to 0 \), condition (2.17) and

\[
\sum_{v=1}^{n-1} t_{vv} |t_{mn}| = O(t_{mn}), \quad n \to \infty \quad (3.22)
\]

where \((X_n)\) is a quasi-f-power increasing sequence, \( f = (f_n) = (n^\beta (\log n)^\gamma) \), where \( \beta, \gamma \) satisfying (2.18), and

\[
\sum_{n=1}^{m} \frac{1}{n X_n^{k-1}} |t_n|^k = O(X_m), \quad m \to \infty, \quad (3.23)
\]
then the series $\sum a_n \lambda_n$ is summable $|T|_k$, $k \geq 1$.

Proof. With the same notations as in the proof of Theorem 3.1, we have

$$Y_n = Y_{n1} + Y_{n2} + Y_{n3} + Y_{n4}.$$ 

In view of (3.22), (1.8), (3.21), (3.23) and (2.20),

$$\begin{align*}
\sum_{n=1}^{m+1} n^{k-1} |Y_{n1}|^k & = \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{v=1}^{n-1} \hat{t}_{nv} \lambda_v t_v \right|^k \\
& \leq \sum_{n=1}^{m+1} n^{k-1} \sum_{v=1}^{n-1} v^{-k} l_{1v}^{1-k} |\hat{t}_{nv}| |\lambda_v| |t_v|^k \left( \sum_{v=1}^{n-1} t_{uv} |\hat{t}_{nv}| \right)^{k-1} \\
& = O(1) \sum_{n=1}^{m+1} n^{k-1} l_{nn}^{1-k} \sum_{v=1}^{n-1} v^{-k} l_{1v}^{1-k} |\hat{t}_{nv}| |\lambda_v| |t_v|^k \\
& = O(1) \sum_{v=1}^{m} v^{-1} |\lambda_v|^k |t_v|^k \\
& = O(1) \sum_{v=1}^{m} \frac{1}{v X_v^{k-1}} |\lambda_v|^k |t_v|^k (|\lambda_v| X_v)^{k-1} \\
& = O(1) \sum_{v=1}^{m} \frac{1}{v X_v^{k-1}} |t_v|^k |\lambda_v| \\
& = O(1) \sum_{v=1}^{m} \frac{1}{r X_r^{k-1}} |t_r|^k \Delta |\lambda_v| + |\lambda_m| \sum_{v=1}^{m} \frac{1}{v X_v^{k-1}} |t_v|^k \\
& = O(1) \sum_{v=1}^{m} X_v |\Delta \lambda_v| + O(1) |\lambda_m| X_m = O(1)
\end{align*}
By (2.15), (1.8) and (2.16),

\[
\sum_{n=1}^{m+1} n^{k-1} |Y_{n2}|^k = \sum_{n=1}^{m+1} n^{k-1} \sum_{v=1}^{n-1} \lambda_v^k \hat{t}_{n,v} \lambda_v^k \left( \sum_{v=1}^{n-1} |\Delta_v \hat{t}_{n,v}| \right)^{k-1}
\]

\[
\leq \sum_{n=1}^{m+1} n^{k-1} \sum_{v=1}^{n-1} |\Delta_v \hat{t}_{n,v}| |\lambda_v|^k |t_v|^k \left( \sum_{v=1}^{n-1} |\Delta_v \hat{t}_{n,v}| \right)^{k-1}
\]

\[
= O(1) \sum_{n=1}^{m+1} \sum_{v=1}^{m+1} |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{m+1} |\Delta_v \hat{t}_{n,v}|
\]

\[
= O(1) \sum_{v=1}^{m} t_{vv} \sum_{n=v+1}^{m+1} |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{m+1} |\Delta_v \hat{t}_{n,v}|
\]

\[
= O(1) \sum_{v=1}^{m} \frac{1}{X_v} |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{m+1} |\Delta_v \hat{t}_{n,v}|
\]

= \(O(1)\sum_{v=1}^{m = 1} \frac{1}{X_v} |\lambda_v|^k |t_v|^k = O(1)\), as in the case \(Y_{n1}\).

By (2.20), (1.8), (2.14) and (2.19),

\[
\sum_{n=1}^{m+1} n^{k-1} |Y_{n3}|^k = \sum_{n=1}^{m+1} n^{k-1} \sum_{v=1}^{n-1} t_{n,v} \hat{t}_{n,v+1} |\Delta \lambda_v|^k \left( \sum_{v=1}^{n-1} X_v |\Delta \lambda_v| \right)^{k-1}
\]

\[
\leq \sum_{n=1}^{m+1} n^{k-1} \sum_{v=1}^{n-1} \frac{1}{X_v^{k-1}} |t_v|^k |\hat{t}_{n,v+1}|^k |\Delta \lambda_v|^k \left( \sum_{v=1}^{n-1} X_v |\Delta \lambda_v| \right)^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m+1} n^{k-1} \sum_{v=1}^{n-1} \frac{1}{X_v^{k-1}} |t_v|^k |\hat{t}_{n,v+1}|^k |\Delta \lambda_v|^k \left( \sum_{v=1}^{n-1} X_v |\Delta \lambda_v| \right)^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} \frac{1}{X_v^{k-1}} |t_v|^k |\Delta \lambda_v|^k \sum_{n=v+1}^{m+1} n^{k-1} |\hat{t}_{n,v+1}|^k
\]
Finally, in view of (1.8),
\[
\sum_{n=1}^{m} n^{k-1} |Y_n|^{k} = \sum_{n=1}^{m} n^{k-1} |\frac{n+1}{m} t_{nn} \lambda_n |^{k} = O(1) \sum_{n=1}^{m} n^{-1} |\lambda_n | |t_n|^{k} = O(1),
\]
as in the case of $Y_{n1}$.

References


