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Absolute Summability Factors Involving Quasi-f-Power Increasing Sequences

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Abstract

Two new general results concerning absolute summability of an infinite series which extended and generalized the result of Sevli and Leindler [4] are achieved.

1 Introduction

Let T be a lower triangular matrix, (s_n) a sequence of the nth partial sums of the series $\sum a_n$, and

$$T_n := \sum_{v=0}^{n} t_{nv} s_v. \tag{1.1}$$

A series $\sum a_n$is said to be summable $|T|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|\Delta T_{n-1}\right|^k < \infty . \tag{1.2}$$

Given any lower triangular matrix T , one can associate the matrices $\bar{T}$ and $\hat{T}$, with entries defined by $\bar{t}_{nv} = \sum_{i=v}^{n} t_{ni}, \quad n, i = 0, 1, 2...,\qquad \hat{t}_{nv} =$

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$\bar{t}_{nv} - \bar{t}_{n-1,v}$ respectively. With $s_n = \sum_{i=0}^{n} a_i\lambda_i$,

$$t_n = \sum_{v=0}^{n} t_{nv}s_v = \sum_{v=0}^{n} t_{nv}\sum_{i=0}^{v} a_i\lambda_i = \sum_{i=0}^{n} a_i\lambda_i \sum_{v=i}^{n} t_{nv} = \sum_{i=0}^{n} \bar{t}_{ni}a_i\lambda_i \,. \quad (1.3)$$

$$Y_n := t_n - t_{n-1} = \sum_{i=0}^{n} \bar{t}_{ni}a_i\lambda_i - \sum_{i=0}^{n-1} \bar{t}_{n-1,i}a_i\lambda_i = \sum_{i=0}^{n} \hat{t}_{ni}a_i\lambda_i, \quad as\ \bar{t}_{n-1,n} = 0. \quad (1.4)$$

Let Ω be the class of all matrices $T = (t_{nv})$ satisfying
(i) T is lower triangular,
(ii) $t_{nv} \geq 0, \quad n, v = 0, 1, ...$,
(iii) $t_{n-1,v} \geq t_{nv}, \quad n \geq v + 1$,
(v) $\vec{t}_{n0} = 1\,, \quad n = 0, 1, ...$.
A positive sequence $\gamma = (\gamma_n)$ is said to be a quasi-β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$Kn^{\beta}\gamma_n \geq m^{\beta}\gamma_m, \quad (1.5)$$

holds for $n \geq m \geq 1$.
A positive sequence $\gamma = (\gamma_n)$ is said to be a quasi-f-power increasing sequence, where $f = (f_n) = (n^{\beta}(\log n)^{\mu}), \ 0 \leq \beta < 1, \ \mu \geq 0$ if (see[5]) there exists a constant $K = K(\gamma, f) \geq 1$ such that

$$Kf_n\gamma_n \geq f_m\gamma_m, \quad (1.6)$$

holds for $n \geq m \geq 1$.
Let t_n denote the nth (C,1) mean of the sequence (na_n), that is

$$t_n = \frac{1}{n+1}\sum_{v=1}^{n} va_v. \quad (1.7)$$

Very recently Sevli and Leindler [4] proved the following result
Theorem 1.1. $T \in \Omega$ *satisfying*

$$nt_{nn} = \mathrm{O}\,(1)\,, \qquad n \to \infty\,, \quad (1.8)$$

and let (λ_n) *be a sequence of real numbers satisfying*

$$\sum_{n=1}^{m} \lambda_n \;=\; o(m), \quad m \to \infty\,, \tag{1.9}$$

$$\sum_{n=1}^{m} |\Delta\lambda_n| \;=\; o(m), \quad m \to \infty. \tag{1.10}$$

If (X_n) *is a quasi-f-power increasing sequence such that the conditions*

$$\sum_{n=1}^{m} \frac{1}{n}\; |t_n|^k \;=\; \mathrm{O}\,(X_m)\,, \qquad m \to \infty, \tag{1.11}$$

$$\sum_{n=1}^{\infty} n\, X_n(\beta,\mu)\; |\Delta\,|\Delta\lambda_n||\; < \infty, \tag{1.12}$$

are satisfied , then the series $\sum a_n\lambda_n$ *is summable* $|T|_k\,,\; k \geq 1,$ *where* $(f_n) \;:=$ $\left(n^{\beta}\,(\log n)^{\mu}\right)\;,\;\; \mu \geq 0,\;\; 0 \leq \beta < 1,$ *and* $X_n(\beta,\mu) \;= max\big(n^{\beta}\,(\log n)^{\mu},\, \log n\big).$

2 Lemmas

Lemma 2.1[1]. *Let* $T \in \Omega\,,$ *then*

$$\hat{t}_{n,v+1} \leq\; t_{nn}\,, \quad n \geq v+1\,, \tag{2.13}$$

and

$$\sum_{n=v+1}^{m+1} \hat{t}_{n,v+1} \;\leq 1\,, \quad n = 0, 1, \dots\; . \tag{2.14}$$

Lemma 2.2[1]. *Let* $T \in \Omega\,,$ *then*

$$\sum_{v=1}^{n-1} \left|\Delta_v \hat{t}_{nv}\right| \leq\; t_{nn}, \tag{2.15}$$

and

$$\sum_{n=v+1}^{m+1} \left|\Delta_v \hat{t}_{nv}\right| \le t_{vv} \, . \tag{2.16}$$

Lemma 2.3. *If (λ_n) is convex sequence such $\sum n^{-1}\lambda_n$is convergent, and the following condition*

$$\sum_{n=1}^{\infty} n^{\beta}(\log n)^{\gamma} X_n \left|\Delta\left|\Delta\lambda_n\right|\right| < \infty, \tag{2.17}$$

where

$$\beta = \begin{cases} 1, & \lambda > 1, \\ > 1, & 0 \le \gamma \le 1, \end{cases} \tag{2.18}$$

holds, then the following inequalities holds

$$nX_n \left|\Delta\lambda_n\right| = \mathrm{O}(1), \quad n \to \infty, \tag{2.19}$$

$$\sum_{n=1}^{\infty} X_n \left|\Delta\lambda_n\right| < \infty \, . \tag{2.20}$$

Proof. Since $\Delta\lambda_n \to 0 \ \ as \ \ n \to \infty$, and $\left(n^{\beta}(\log n)^{\gamma} X_n\right)$ is non-decreasing, $\sum_{n=1}^{\infty} X_n \left|\Delta\lambda_n\right| = \sum_{n=1}^{\infty} X_n \sum_{v=n}^{\infty} \Delta\left|\Delta\lambda_v\right| \le \sum_{n=1}^{\infty} X_n \sum_{v=n}^{\infty} \left|\Delta\left|\Delta\lambda_v\right|\right| =$ $\sum_{v=1}^{\infty} \left|\Delta\left|\Delta\lambda_v\right|\right| \sum_{n=1}^{v} X_n = \sum_{v=1}^{\infty} \left|\Delta\left|\Delta\lambda_v\right|\right| \sum_{n=1}^{v} X_n n^{\beta}(\log n)^{\gamma} n^{-\beta}(\log n)^{-\gamma} =$ $\mathrm{O}(1) \sum_{v=1}^{\infty} v^{\beta}(\log v)^{\gamma} X_v \left|\Delta\left|\Delta\lambda_v\right|\right| \sum_{n=1}^{v} n^{-\beta}(\log n)^{-\gamma} =$ $\mathrm{O}(1) \sum_{v=1}^{\infty} v^{\beta}(\log v)^{\gamma} X_v \left|\Delta\left|\Delta\lambda_v\right|\right| \sum_{n=1}^{\infty} \frac{1}{n^{\beta}(\log n)^{\gamma}} = \mathrm{O}(1) \sum_{v=1}^{\infty} v^{\beta}(\log v)^{\gamma} X_v \left|\Delta\left|\Delta\lambda_v\right|\right| =$ $\mathrm{O}(1)$. $nX \left|\Delta\lambda_n\right| \le nX_n \sum_{v=n}^{\infty} \left|\Delta\left|\Delta\lambda_{\nu}\right|\right| = \mathrm{O}(1) n^{\beta}(\log n)^{\gamma} X_n \sum_{v=n}^{\infty} \left|\Delta\left|\Delta\lambda_{\nu}\right|\right| =$ $\mathrm{O}(1) \sum_{v=n}^{\infty} v^{\beta}(\log v)^{\gamma} X_v \left|\Delta\left|\Delta\lambda_{\nu}\right|\right| = \mathrm{O}(1) \ \sum_{v=1}^{\infty} v^{\beta}(\log v)^{\gamma} X_v \left|\Delta\left|\Delta\lambda_{\nu}\right|\right| = \mathrm{O}(1)$.

3 Main result

The object of this paper is to give two general results in both we are extending the range of β. Theorem 1.1 is dealing with all β such that $0 \le \beta < 1$. Our results are dealing with all β such that $\beta \ge 1$.

Theorem 3.1. Let $T \in \Omega$ *satisfy* (1.8) *and let* (λ_n) *be a sequence of bounded variation such that* $\sum n^{-1}\lambda_n$ *is convergent. Let* (1.11) *and* (2.17) *be satisfied and*

$$|\lambda_n| \, X_n \; = \; \mathrm{O}\,(1), \quad as\, n \to \infty, \tag{3.21}$$

where (X_n) *is a quasi-*f*-power increasing sequence,* $f = (f_n) = (\, n^{\beta}(\log n)^{\gamma})$*, where* $\beta,\ \gamma$ *satisfying* (2.18) *, then the series* $\sum a_n\lambda_n$*is summable* $|T|_k\,,\ k \geq 1\,.$
Proof. By Abel's transformation, we have

$$Y_n \; = \; \sum_{v=`1}^{n} \hat{t}_{nv} a_v \lambda_v \; = \; \sum_{v=1}^{n} v a_v v^{-1} t_{nv} \lambda_v$$

$$= \; \sum_{v=1}^{n-1} \left(\sum_{r=1}^{v} r a_r \right) \Delta_v (v^{-1} \hat{t}_{nv} \lambda_v) \; + \; n^{-1} t_{nn} \lambda_n \sum_{v=1}^{n} v a_v$$

$$= \sum_{v=1}^{n-1} (v+1) t_v \left(\frac{1}{v(v+1)} \hat{t}_{nv} \lambda_v + \frac{1}{(v+1)} \Delta_v \hat{t}_{nv} \lambda_v + \frac{1}{(v+1)} \hat{t}_{n,v+1} \Delta \lambda_v \right)$$

$$+ \frac{n+1}{n} \, t_{nn} \lambda_n t_n$$

$$= \sum_{v=1}^{n-1} v^{-1} \hat{t}_{nv} \lambda_v t_v \; + \sum_{v=1}^{n-1} \Delta_v \hat{t}_{nv} \lambda_v t_v \; + \sum_{v=1}^{n-1} t_v \hat{t}_{n,v+1} \Delta \lambda_v \; + \; \frac{n+1}{n} \, t_{nn} \lambda_n t_n$$

$$= \; Y_{n1} + Y_{n2} + Y_{n3} + Y_{n4}.$$

In order to complete the proof of the theorem, by Minkowski's inequality, it is therefore sufficient to show that $\sum_{n=1}^{m} \alpha_n \left|Y_{nj}\right|^k < \infty \ j = 1, 2, 3, 4$. Applying Hölder's inequality, we have, via (3.21), (1.8), (2.14) and (2.20) ,

$$\sum_{n=1}^{m+1} n^{k-1} \left|Y_{n1}\right|^k = \; \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{v=1}^{n-1} v^{-1} \hat{t}_{nv} \lambda_v t_v \right|^k$$

$$\leq \; \sum_{n=1}^{m+1} n^{k-1} \sum_{v=1}^{n-1} v^{-1} \left|\hat{t}_{nv}\right|^k \, |\lambda_v| \, |t_v|^k \left(\sum_{v=1}^{n-1} v^{-1} \, |\lambda_v| \right)^{k-1}$$

$$= \mathrm{O}(1) \sum_{n=1}^{m} n^{k-1} \left|\hat{t}_{nv}\right|^{k} \sum_{v=1}^{n-1} v^{-1} \left|\lambda_v\right| \left|t_v\right|^k$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} v^{-1} \left|\lambda_v\right| \left|t_v\right|^k \sum_{n=v+1}^{m+1} n^{k-1} \left|t_{nn}\right|^{k-1} \left|\hat{t}_{nv}\right|$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} v^{-1} \left|\lambda_v\right| \left|t_v\right|^k \sum_{n=v+1}^{m+1} \left|\hat{t}_{nv}\right| = \mathrm{O}(1) \sum_{v=1}^{m} v^{-1} \left|\lambda_v\right| \left|t_v\right|^k$$

$$= \mathrm{O}(1) \sum_{v=1}^{m-1} \left(\sum_{r=1}^{v} r^{-1} \left|t_r\right|^k\right) \Delta \left|\lambda_v\right| + \mathrm{O}(1) \left|\lambda_m\right| \sum_{v=1}^{m} v^{-1} \left|t_v\right|^k$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} X_v \left|\Delta\lambda_v\right| + \mathrm{O}(1) \left|\lambda_m\right| X_m = \mathrm{O}(1).$$

By (2.15), (1.8) and (2.16), $\sum_{n=1}^{m+1} n^{k-1} \left|Y_{n2}\right|^k = \sum_{n=1}^{m+1} n^{k-1} \left|\sum_{v=1}^{n-1} \Delta_v \hat{t}_{nv} \lambda_v t_v\right|^k \leq \sum_{n=1}^{m+1} n^{k-1} \sum_{v=1}^{n-1} \left|\Delta_v \hat{t}_{nv}\right| \left|\lambda_v\right|^k \left|t_v\right|^k \left(\sum_{v=1}^{n-1} \left|\Delta_v \hat{t}_{nv}\right|\right)^{k-1} =$ $\mathrm{O}(1) \sum_{n=1}^{m+1} n^{k-1} (t_{nn})^{k-1} \sum_{v=1}^{n-1} \left|\Delta_v \hat{t}_{nv}\right| \left|\lambda_v\right|^k \left|t_v\right|^k =$ $\mathrm{O}(1) \sum_{v=1}^{m} \left|\lambda_v\right|^k \left|t_v\right|^k \sum_{n=v+1}^{m+1} (n t_{nn})^{k-1} \left|\Delta_v \hat{t}_{nv}\right| = \mathrm{O}(1) \sum_{v=1}^{m} \left|\lambda_v\right|^k \left|t_v\right|^k \sum_{n=v+1}^{m+1} \left|\Delta_v \hat{t}_{nv}\right| =$ $\mathrm{O}(1) \sum_{v=1}^{m} t_{vv} \left|\lambda_v\right|^k \left|t_v\right|^k = \mathrm{O}(1) \sum_{v=1}^{m} v^{-1} \left|\lambda_v\right|^k \left|t_v\right|^k = \mathrm{O}(1)$, as in the case Y_{n1}.

By (1.8) and (2.14) ,

$$\sum_{n=1}^{m+1} n^{k-1} \left|Y_{n3}\right|^k = \sum_{n=1}^{m+1} n^{k-1} \left|\sum_{v=1}^{n-1} t_v \hat{t}_{n,v+1} \Delta\lambda_v\right|^k$$

$$\leq \sum_{n=1}^{m+1} n^{k-1} \sum_{v=1}^{n-1} \left|t_v\right|^k \left|\hat{t}_{n,v+1}\right|^k \left|\Delta\lambda_v\right| \left(\sum_{v=1}^{n-1} \left|\Delta\lambda_v\right|\right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} |t_v|^k |\Delta\lambda_v| \sum_{n=v+1}^{m+1} n^{k-1} |t_{n,v+1}|^k$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} |t_v|^k |\Delta\lambda_v| \sum_{n=v+1}^{m+1} (nt_{nn})^{k-1} \left|\hat{t}_{n,v+1}\right|$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} |t_v|^k |\Delta\lambda_v| \sum_{n=v+1}^{m+1} \left|\hat{t}_{n,v+1}\right|$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} |t_v|^k |\Delta\lambda_v| = \mathrm{O}(1) \sum_{v=1}^{m} v^{-1} |t_v|^k \, v \, |\Delta\lambda_v|$$

$$= \mathrm{O}(1) \sum_{v=1}^{m-1} \left(\sum_{r=1}^{v} r^{-1} |t_r|^k \right) \Delta \left(v \, |\Delta\lambda_v| \right) + \left(\sum_{v=1}^{m} v^{-1} |t_v|^k \right) m \, |\Delta\lambda_m|$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} X_v \, |\Delta\lambda_v| + \mathrm{O}(1) \sum_{v=1}^{m} v X_v \, |\Delta \, |\Delta\lambda_v|| + \mathrm{O}(1) \, m \, X_m \, |\Delta\lambda_m| = \mathrm{O}(1).$$

Finally, by (1.8),

$$\sum_{n=1}^{m} n^{k-1} |Y_{n4}|^k = \sum_{n=1}^{m} n^{k-1} \left| \frac{n+1}{n} \, t_{nn} \lambda_n t_n \right|^k$$

$= \mathrm{O}(1) \sum_{n=1}^{m} (nt_{nn})^{k-1} t_{nn} |\lambda_n|^k |t_n|^k = \mathrm{O}(1) \sum_{n=1}^{m} n^{-1} |\lambda_n| \, |t_n|^k = \mathrm{O}(1)$, as in the case of Y_{n1}.

Theorem 3.2. **Let** $T \in \Omega$ *satisfy* (1.8) *and let* (λ_n) *be a sequence of real numbers satisfying* $\Delta\lambda_n \to 0$ *,condition* (2.17) *and*

$$\sum_{v=1}^{n-1} t_{vv} \left|\hat{t}_{nv}\right| = \mathrm{O}(t_{nn}), \quad n \to \infty \tag{3.22}$$

where (X_n) *is a quasi-*f*-power increasing sequence,* $f = (f_n) = (n^\beta (\log n)^\gamma)$*, where* $\beta, \ \gamma$ *satisfying* (2.18), *and*

$$\sum_{n=1}^{m} \frac{1}{n X_n^{k-1}} \, |t_n|^k = \mathrm{O}(X_m), \qquad m \to \infty, \tag{3.23}$$

then the series $\sum a_n \lambda_n$ *is summable* $|T|_k\,,\ k \geq 1\,.$

Proof. With the same notations as in the proof of Theorem 3.1, we have

$$Y_n \;=\; Y_{n1} + Y_{n2} + Y_{n3} + Y_{n4}\,.$$

In view of (3.22), (1.8), (3.21) , (3.23) and (2.20),

$$\sum_{n=1}^{m+1} n^{k-1}\left|Y_{n1}\right|^k = \sum_{n=1}^{m+1} n^{k-1}\left|\sum_{v=1}^{n-1} v^{-1}\hat{t}_{nv}\lambda_v t_v\right|^k$$

$$\leq \sum_{n=1}^{m+1} n^{k-1}\sum_{v=1}^{n-1} v^{-k} t_{vv}^{1-k}\left|\hat{t}_{nv}\right|\left|\lambda_v\right|\left|t_v\right|^k\left(\sum_{v=1}^{n-1} t_{vv}\left|\hat{t}_{nv}\right|\right)^{k-1}$$

$$= \mathrm{O}\,(1)\sum_{n=1}^{m+1} n^{k-1} t_{nn}^{k-1}\sum_{v=1}^{n-1} v^{-k} t_{vv}^{1-k}\left|\hat{t}_{nv}\right|\left|\lambda_v\right|\left|t_v\right|^k$$

$$= \mathrm{O}\,(1)\sum_{v=1}^{m} v^{-k} t_{vv}^{1-k}\left|\lambda_v\right|^k\left|t_v\right|^k\sum_{v=1}^{n-1} n^{k-1} t_{nn}^{k-1}\left|\hat{t}_{nv}\right|$$

$$= \mathrm{O}\,(1)\sum_{v=1}^{m} v^{-1}\left|\lambda_v\right|^k\left|t_v\right|^k$$

$$= \mathrm{O}\,(1)\sum_{v=1}^{m}\frac{1}{vX_v^{k-1}}\left|\lambda_v\right|\left|t_v\right|^k\left(\left|\lambda_v\right| X_v\right)^{k-1}$$

$$= \mathrm{O}\,(1)\sum_{v=1}^{m}\frac{1}{vX_v^{k-1}}\left|t_v\right|^k\left|\lambda_v\right|$$

$$= \mathrm{O}\,(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\frac{1}{rX_r^{k-1}}\left|t_r\right|^k\right)\Delta\left|\lambda_v\right| + \left|\lambda_m\right|\sum_{v=1}^{m}\frac{1}{vX_v^{k-1}}\left|t_v\right|^k$$

$$= \mathrm{O}\,(1)\sum_{v=1}^{m} X_v\left|\Delta\lambda_v\right| + \mathrm{O}\,(1)\left|\lambda_m\right| X_m = \mathrm{O}\,(1)$$

By (2.15), (1.8) and (2.16),

$$\sum_{n=1}^{m+1} n^{k-1} |Y_{n2}|^k = \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{v=1}^{n-1} \Delta_v \hat{t}_{nv} \lambda_v t_v \right|^k$$

$$\leq \sum_{n=1}^{m+1} n^{k-1} \sum_{v=1}^{n-1} \left| \Delta_v \hat{t}_{nv} \right| |\lambda_v|^k \, |t_v|^k \left(\sum_{v=1}^{n-1} \left| \Delta_v \hat{t}_{nv} \right| \right)^{k-1}$$

$$= \mathrm{O}(1) \sum_{n=1}^{m+1} n^{k-1} (t_{nn})^{k-1} \sum_{v=1}^{n-1} \left| \Delta_v \hat{t}_{nv} \right| |\lambda_v|^k \, |t_v|^k$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} |\lambda_v|^k \, |t_v|^k \sum_{n=v+1}^{m+1} (n t_{nn})^{k-1} \left| \Delta_v \hat{t}_{nv} \right|$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} |\lambda_v|^k \, |t_v|^k \sum_{n=v+1}^{m+1} \left| \Delta_v \hat{t}_{nv} \right|$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \frac{t_{vv}}{X_v^{k-1}} |\lambda_v| \, |t_v|^k \left(|\lambda_v| \, X_v \right)^{k-1}$$

$= \mathrm{O}(1) \sum_{v=1}^{m} \frac{1}{v X_v^{k-1}} |\lambda_v| \, |t_v|^k = \mathrm{O}(1)$, as in the case Y_{n1}.
By (2.20), (1.8), (2.14) and (2.19),

$$\sum_{n=1}^{m+1} n^{k-1} |Y_{n3}|^k = \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{v=1}^{n-1} t_v \hat{t}_{n,v+1} \Delta\lambda_v \right|^k$$

$$\leq \sum_{n=1}^{m+1} n^{k-1} \sum_{v=1}^{n-1} \frac{1}{X_v^{k-1}} |t_v|^k \left| \hat{t}_{n,v+1} \right|^k |\Delta\lambda_v| \left(\sum_{v=1}^{n-1} X_v \, |\Delta\lambda_v| \right)^{k-1}$$

$$= \mathrm{O}(1) \sum_{n=1}^{m+1} n^{k-1} \sum_{v=1}^{n-1} \frac{1}{X_v^{k-1}} |t_v|^k \left| \hat{t}_{n,v+1} \right|^k |\Delta\lambda_v|$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \frac{1}{X_v^{k-1}} |t_v|^k \, |\Delta\lambda_v| \sum_{n=v+1}^{m+1} n^{k-1} \left| \hat{t}_{n,v+1} \right|^k$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \frac{1}{X_v^{k-1}} |t_v|^k |\Delta\lambda_v| \sum_{n=v+1}^{m+1} n^{k-1} t_{nn}^{k-1} \left|\hat{t}_{n,v+1}\right|$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \frac{1}{vX_v^{k-1}} |t_v|^k v |\Delta\lambda_v|$$

$$= \mathrm{O}(1) \sum_{v=1}^{m-1} \left(\sum_{r=1}^{v} \frac{1}{rX_r^{k-1}} |t_r|^k \right) \Delta(v|\Delta\lambda_v|) + \left(\sum_{v=1}^{m} \frac{1}{vX_v^{k-1}} |t_v|^k \right) m |\Delta\lambda_m|$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} X_v |\Delta\lambda_v| + \mathrm{O}(1) \sum_{v=1}^{m} vX_v |\Delta|\Delta\lambda_v|| + \mathrm{O}(1)\, m\, X_m |\Delta\lambda_m| = \mathrm{O}(1).$$

Finally, in view of (1.8), $\sum_{n=1}^{m} n^{k-1} |Y_{n4}|^k = \sum_{n=1}^{m} n^{k-1} \left|\frac{n+1}{n} t_{nn}\lambda_n t_n\right|^k = \mathrm{O}(1) \sum_{n=1}^{m} (nt_{nn})^{k-1} t_{nn} |\lambda_n|^k |t_n|^k = \mathrm{O}(1) \sum_{n=1}^{m} n^{-1} |\lambda_n| |t_n|^k = \mathrm{O}(1)$, as in the case of Y_{n1}.

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