International Journal of Mathematics and
Computer Science, **7**(2012), no. 2, 101–112

Subsets of Prime Numbers

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(Received February 10, 2012, Accepted October 14, 2012)

Abstract

The Fundamental Theorem of Arithmetic shows the importance of prime numbers. A well-known result is that the set of prime numbers is infinite (the subset of even prime numbers is obviously finite while that of odd prime numbers is therefore infinite). The subset of Ramanujan primes is infinite. The set of triplet prime numbers is finite while it is not known whether or not the subset of twin prime numbers is infinite even though it is so conjectured. We give many results involving the different types of prime numbers.

1 Introduction

A **prime triplet** is a triplet of the form $(p, p+2, p+4)$ consisting of three primes. The set of prime triplets is finite. Indeed, it is a singleton set containing only $(3, 5, 7)$. To see this, notice that $(3, 5, 7)$ is a prime triplet. To prove it is the only one, we use the contradiction method. Suppose there was a prime triplet $(P, P+2, P+4)$ with $P \neq 3$. Then P can be written as $3k, 3k+1$ or $3k+2$. for some integer $k = 2, 3, 4, ...$ However,
If $P = 3k$, then P is not prime.
If $P = 3k+1$, then $P+2$ is not prime being $3(k+1)$.

Key words and phrases: Prime numbers, Ramanujan Primes, Bertrand Postulate, Twin Prime Conjecture, Goldbach Conjecture, Different kinds of primes, Riemann Hypothesis.
AMS (MOS) Subject Classifications: 11M06, 11M38, 11M50.

If $P = 3k + 2$, then $P + 4$ is not prime being $3(k + 2)$.

In 1845 Bertrand [3] conjectured that if an integer $n \geq 1$, then there is at least one prime number p such that $n < p \leq 2n$. Due to its elegance, this was proven at different times by many prominent mathematicians including Tschebyschef [8] in 1850, Landau [7] in 1909, Ramanujan [11] in 1919, and Erdös [9] in 1932. In contrast to Landau's and Tschebyschef's analytic proofs, Erdös gave an elementary proof (it is worth mentioning here that this was not only Erdös first result but he proved it when he was only 18 years old). As we shall see in the next section Ramanujan proved an extension of Bertrand Postulate.

2 Ramanujan extension of Bertrand Postulate. Ramanujan Primes

Theorem 2.1. (Ramanujan) *Let $\pi(x)$ denote the number of primes $\leq x$. Then if*

$$x \geq 2, 11, 17, 29, 41, 47, 59, 67, 71, 97, 101, 107, 127, 149, 151, 167, ...$$

we have the respective results

$$\pi(x) - \pi(\frac{1}{2}x) \geq 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...$$

Remark 2.2. *In particular, the first case is interesting being Bertrand Postulate:*
If $x \geq 2$, then $\pi(x) - \pi(\frac{1}{2}x) \geq 1$.
To see this, simply let $x = 2n$.

The "converse" of Ramanujan Theorem motivates the following definition of Ramanujan Primes the first few of which are
$2, 11, 17, 29, 41, 47, 59, 67, 71, 97, 101, 107, 127, 149, 151, 167$:

Definition 2.3. *For a natural number n, the n^{th} Ramanujan Prime is the smallest integer R_n such that if $x \geq R_n$, then $\pi(x) - \pi(\frac{1}{2}x) \geq n$. That is, whenever $x \geq R_n$, there exist prime numbers between $\frac{x}{2}$ and x. The smallest such R_n must be a prime number as the function $\pi(x) - \pi(\frac{1}{2}x)$ can increase only at a prime.*
This easily implies that there are infinitely many Ramanujan Primes.

Note that Bertrand Postulate is $R_1 = 2$.
Recall that if p *and* $p+2$ *are prime numbers, then this pair is said to be a Twin Prime Pair. The smallest such pair is* $(3, 5)$. *The Twin Prime Conjecture states that there are infinitely many Twin Prime Pairs. Similarly, we define a Twin Ramanujan Prime Pair. The smallest such pair is* $(149, 151)$ *as we can see from the beginning of this section. As we might expect there are fewer Twin Ramanujan Prime Pairs than Twin Prime pairs. As a matter of fact, among the first* 1100 *prime numbers, there are* 70 *Twin Ramanujan Prime Pairs and* 186 *Twin Prime Pairs. Note that* $\frac{70}{186} > \frac{1}{4}$. *This observation is important for the next section.*

3 Results

The following conjecture was stated in [12]:
If $x \geq 571$, then the number of pairs of Twin Ramanujan Primes not exceeding x is greater than one quarter of the number of pairs of Twin Primes not exceeding x. In particular, if there are infinitely many Twin Primes, then there are also infinitely many Twin Ramanujan Primes.
On the other hand, if there are finitely many Twin Ramanujan Primes, then there are finitely many twin primes. Therefore, if Sondow Conjecture is proven and if it is proven also that there are only finitely many Twin Ramanujan Primes, then that would settle the famous Twin Prime Conjecture.

In a letter to Leonhard Euler in 1742, Christian Goldbach stated that he believes that:

G: Every integer greater than 5 is the sum of three primes.

Euler replied:

E: "That every even number ≥ 4 is a sum of two primes, I consider an entirely certain theorem in spite of that I am not able to demonstrate it".

As a matter of fact, if (G) is satisfied and $2n \geq 4$, then $2n + 2 = p_1 + p_2 + p_3$ with primes p_1, p_2, and p_3 that cannot all be odd, for otherwise their sum would be odd. So at least one of them, say p_3, is 2. Then (E) obtains.

Conversely, if (E) is satisfied and $k > 5$, then $2k \geq 6$ and we consider two cases:

Case 1: k is an even integer. $k = 2n, say$. Then $k - 2 = 2n - 2 \geq 4$, and so $k - 2 = p_1 + p_2$, with primes p_1 and p_2. Consequently, $k = p_1 + p_2 + 2$ and so (G) obtains in this case.

Case 2: k is an odd integer. $k = 2n + 1, say$. Then by case 1 applied to $2n$ we get $k = p_1 + p_2 + 3$ and so (G) also obtains in this case.

Notice that (E) is true for infinitely many even integers of the form $2p$ where p is a prime number (since $2p = p + p$ and the set of primes is infinite). But the question remains whether (E) is true for all even integers ≥ 4. Until today, this has not been proved and is known as the **Goldbach conjecture.** The best result in conjunction with this conjecture is by Jing-Run Chen [1] who proved in 1966 (announcing results, but the detailed proofs appeared in two parts one in 1973 and the other in 1978) that from some large number on, every even number is a sum of a prime and a number that is either a prime or a product of at most two primes.
The Goldbach Conjecture crept into the news as a million-dollar question. Indeed, as a publicity stunt, British publisher Tony Faber announced the prize for the first person to prove the Goldbach Conjecture before an appearance of *Uncle Petros and Goldbach's Conjecture,* a fiction novel by the Greek writer Apostolos Doxiados centering on a man devoting his life to find a proof of the Goldbach Conjecture. No one succeeded but it is worth mentioning that the novel was a huge success and got translated into 15 languages.

The following theorem provides an equivalent statement to Goldbach conjecture:

Theorem 3.1. *For each* $n \in \mathbb{N}$, *there is at least one* $k \in \mathbb{N}$ *such that* $n + k$ *and* $(n+2) - k$ *are prime numbers if and only if every integer* ≥ 4 *is the sum of two primes.*

Proof. ($\Rightarrow$): Let $j \geq 4$ be an even integer. Write $j = 2(n + 1)$ where $n \in \mathbb{N}$. Then there is at least one $k \in \mathbb{N}$ such that $n + k$ and $(n + 2) - k$ are prime numbers with

$$j = 2(n + 1) = [n + k] + [(2 + n) - k].$$

($\Leftarrow$): Let $n \in \mathbb{N}$. Clearly $2(n + 1)$ is an even integer ≥ 4. Then there are primes p and q such that

$$2n + 2 = p + q.$$

Therefore for all $k \in \mathbb{N}$ we have

$$[n+k] + [(n+2) - k] = p + q.$$

Since $2n + 2 = p + q$, we can now choose $k \in \mathbb{N}$ such that $n + k = p$ and $(n+2) - k = q$.

In addition, the following theorem provides an equivalent statement to the Twin Prime conjecture:

Theorem 3.2. *For each $n \in \mathbb{N}$, there is at least one $k \in \mathbb{N}$ such that $n + k$ and $(n+2) + k$ are prime numbers if and only if there are infinitely many pairs $(p, p+2)$ such that p and $p+2$ are both prime numbers.*

Proof. ($\Rightarrow$): This follows since $\mathbb{N}$ is infinite and the representation

$$[(2+n) + k] - [n+k] = 2$$

holds with $n + k$ and $(2+n) + k$ prime numbers (Notice that distinct values n_1 and n_2 in $\mathbb{N}$ yield distinct values of $k \in \mathbb{N}$, for if there is $k \in \mathbb{N}$ such that $(n_1 + k, n_1 + k + 2) = (n_2 + k, n_2 + k + 2)$, then $n_1 = n_2$.)
($\Leftarrow$): Let $n \in \mathbb{N}$. Using the hypothesis, we can choose $k \in \mathbb{N}$ such that $n + k$ and $(n+2) + k$ are prime numbers.

Remark 3.3. *Looking at the statements on the left-hand sides in the previous two theorems, we observe the unusual feature that proving one of the Goldbach or the Twin Prime conjecture proves the other.*

4 Connection between various kinds of primes

With $\pi(x)$ denoting the number of primes $\leq x$ and p_n denoting the nth prime number, let us state and prove the following results

Lemma 4.1. $\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1 \Leftrightarrow \lim_{n\to\infty} \frac{p_n}{n \log n} = 1.$

Proof. ($\Rightarrow$) : Let $x = p_n$. Then $\lim_{n\to\infty} \frac{\pi(p_n)}{p_n/\log p_n} = 1$. Since $\pi(p_n) = n, \lim_{n\to\infty} \frac{n}{p_n/\log p_n} = 1$. Then there is $n_0 \in N$ such that

$$n \geq n_0 \Rightarrow n \geq \frac{1}{2}\frac{p_n}{\log p_n}.$$

Then

$$n \geq n_0 \Rightarrow p_n \leq 2n \log p_n.$$

Therefore

$$n \geq n_0 \Rightarrow \log p_n \leq \log 2 + \log n + \log\log p_n.$$

Hence

$$n \geq n_0 \Rightarrow \log p_n (1 - \frac{\log\log p_n}{\log p_n}) \leq \log 2 + \log n$$

or

$$n \geq n_0 \Rightarrow \frac{\log p_n}{\log n}(1 - \frac{\log\log p_n}{\log p_n}) \leq \frac{\log 2}{\log n} + 1.$$

Now, since $\lim_{n\to\infty} \frac{\log\log p_n}{\log p_n} = 0$ and $\lim_{n\to\infty} \frac{\log 2}{\log n}$, given $\epsilon > 0$, there exists $n_1 \in N$ such that

$$n \geq n_1 \Rightarrow 1 - \epsilon \leq \frac{\log p_n}{\log n} \leq 1 + \epsilon.$$

Since ϵ was arbitrary it follows that

$$\lim_{n\to\infty} \frac{\log p_n}{\log n} = 1.$$

Using this we can now write

$$\lim_{n\to\infty} \frac{n}{p_n / \log n} = 1.$$

That is,

$$\lim_{n\to\infty} \frac{p_n}{n \log n} = 1.$$

$(\Leftarrow)$: For any x, there exist prime numbers p_n and p_{n+1} such that $p_n \leq x < p_{n+1}$. Hence $\pi(x) = n$. Since the function $f(x) = \frac{x}{\log x}$ is increasing,

$$\frac{p_n}{\log p_n} \leq \frac{x}{\log x} \leq \frac{p_{n+1}}{\log p_{n+1}}.$$

Multiplying by $\frac{1}{n}$ we get

$$\frac{p_n}{n \log p_n} \leq \frac{x}{\pi(x) \log x} \leq \frac{p_{n+1}}{n \log p_{n+1}}.$$

Taking limits as $n \to \infty$ (and hence $x \to \infty$) and using the hypothesis, the result follows and the proof of the lemma is complete.

Remark 4.2. *The Prime Number Theorem states that*

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1.$$

By the above lemma it follows that the Prime Number Theorem is equivalent to $p_n \sim n\log n, n \to \infty$.

Definition 4.3. *(a) A set* S *in a vector space is said to be* **convex** *if whenever* $x, y \in S$ *and* $0 < \lambda < 1, (1-\lambda)x + \lambda y \in S$.
(b) The **convex hull** *or* **convex envelope** *of a set of points* S *in* n *dimensions is the intersection of all convex sets containing* S.

Theorem 4.4. *[10] Let* $0 < a_1 < a_2 < \ldots$ *be a sequence of numbers with* $\lim_{n\to\infty} \frac{n}{a_n} = 0$. *Then there are infinitely many* n *for which* $2a_n < a_{n-i} + a_{n+i}$ *for all positive* $i < n$.

Proof. Clearly the boundary of the convex hull of the set $\{(n, a_n) : n = 1, 2, \ldots\}$ is polygonal. The additional assumption $\lim_{n\to\infty} \frac{n}{a_n} = 0$ implies that the nonvertical portion of this polygonal boundary is convex and has infinitely many vertices. Since each of these vertices is of the form (n, a_n) for some n, the result follows.

Corollary 4.5. *There are infinitely many* n *for which* $2p_n < p_{n-i} + p_{n+i}$ *for all positive* $i < n$.

Proof. Clearly $0 < p_1 < p_2 < \ldots$ In addition, using the above lemma, it follows that $\lim_{n\to\infty} \frac{n}{p_n} = 0$.

Theorem 4.6. *Let* $0 < a_1 < a_2 < \ldots$ *be a sequence of numbers with* $\lim_{n\to\infty} \frac{a_n}{n} = 0$. *Then there are infinitely many* n *for which* $2a_n > a_{n-i} + a_{n+i}$ *for all positive* $i < n$.

Proof. Clearly the boundary of the convex hull of the set $\{(n, a_n) : n = 1, 2, \ldots\}$ is polygonal. The additional assumption $\lim_{n\to\infty} \frac{a_n}{n} = 0$ implies that the nonhorizontal portion of this polygonal boundary is concave and has infinitely many vertices. Since each of these vertices is of the form (n, a_n) for some n, the result follows.

Remark 4.7. *Since* $\lim_{n\to\infty} \frac{p_n}{n}$ *is not necessarily* 0, *we cannot hope to get that there are infinitely many* n *for which* $2p_n > p_{n-i} + p_{n+i}$ *for all positive* $i < n$ *from the above theorem.*

However, we can prove the following

Corollary 4.8. *There are infinitely many n for which $p_n^2 > p_{n-i}p_{n+i}$ for all positive $i < n$.*

Proof. Clearly $0 < \log p_1 < \log p_2 < \ldots$ In addition, using the above lemma, we have $p_n \sim n \log n$ as $n \to \infty$ and so $\log p_n \sim \log n + \log \log n$ as $n \to \infty$ from which it follows that $\lim_{n\to\infty} \frac{\log p_n}{n} = 0$. Thus there are infinitely many n such that $2 \log p_n > \log p_{n-i} + \log p_{n+i}$ for all positive $i < n$. Upon exponentiation we get our result.

Definition 4.9. *A* **balanced prime number** *is a prime which is the average of the nearest primes surrounding it. That is, a balanced prime p_n satisfies the equality $p_n = \frac{p_{n-1}+p_{n+1}}{2}$.*

The first few balanced primes are
$5, 53, 157, 173, 211, 257, 263, 373, 563, 593, 607, 653, 733, 947, 977, 1103$.
It is conjectured that there are infinitely many balanced primes.

Definition 4.10. *A related definition is that of a* **strong prime number (weak prime number)** *which is greater (less) than the average.*

The first strong primes are:
$11, 17, 29, 37, 41, 59, 67, 71, 79, 97, 101, 107, 127, 137, 149, 163, 179, 191, 197, 223, 227, 239, 251, 269, 277, 281, 307, 311, 331, 347, 367, 379, 397, 419, 431, 439, 457, 461, 479, 487, 499$.
The first weak primes are:
$3, 7, 13, 19, 23, 31, 43, 47, 61, 73, 83, 89, 103, 109, 113, 131, 139, 151, 167, 181, 193, 199, 229, 233, 241, 271, 283$.
Note that the number of strong primes and the number of weak primes are equal (26) for the first time at the prime $P_{60} = 281$. It follows from a particular case of corollary 4.5 that the number of weak primes is infinite. But, unfortunately, we cannot say the same thing for strong prime numbers at least using the above results and the question remains open. However, in a twin prime pair $(p, p+2)$ with $p > 5, p$ is always a strong prime number because 3 must divide $p-2$ which cannot be prime. Consequently, if the twin prime conjecture is true, then the number of strong primes is infinite.

Definition 4.11. *A* **good prime number** *is a prime whose square exceeds the product of any two primes at the same number of positions before and after it in the sequence of primes; that is, a good prime p_n satisfies the inequality $p_n^2 > p_{n-i}p_{n+i}$ for $1 \le i \le n-1$.*

The first good primes are $5, 11, 17, 29, 37, 41, 53, 59, 67, 71, 97, 101, 127, 149$. It follows from a particular case of corollary 4.8 that the number of good primes is infinite.

Definition 4.12. *We now define* **sexy prime numbers**. *Those are pairs of primes differing by* 6 *(sex is a Latin word for six).*

Obviously one-digit sexy pairs of primes do not exist.
There are seven 2-digit sexy prime pairs:

$$(23, 29), (31, 37), (47, 53), (53, 59), (61, 67), (73, 79), (83, 89).$$

Remark 4.13. *Notice the similarity to twin primes. As in the case of twin primes, it is not known whether or not there are infinitely many sexy primes.*

Definition 4.14. *Naturally, a* **Chen prime number** *is a prime p such that $p + 2$ is either a prime or a product of two primes.*

Remark 4.15. *Jing-Run Chen himself [1] proved that there are infinitely many Chen primes.*

Definition 4.16. *A* **permutable prime number** *is a prime number which remains prime in all permutations of its digits.*

All the permutable primes with less than 49081 digits are:
$2, 3, 5, 7, 11, 13, 17, 31, 37, 71, 73, 79, 97, 113, 131, 199, 311, 337, 373, 733, 919, 991,$
$1111111111111111111, 11111111111111111111111, R_{317}, R_{1031}$, where $R_n = \frac{10^n-1}{9}$
is the number with n ones.
I conjecture that the number of permutable primes is finite.

Definition 4.17. *A* **pseudo-prime permutable number** *is a number where at least a permutable portion of it is prime.*

Clearly, the number of pseudo-prime permutable numbers is infinite because, for instance, $2, 22, 222, \cdots$ are pseudo-prime permutable numbers.

Definition 4.18. *A* **Fibonacci prime number** *is a Fibonacci number that is prime.*

The first Fibonacci primes are
$2, 3, 5, 13, 89, 233, 1597, 28657, 514229, 433494437, 2971215073$
and, as expected, it is not known if there are infinitely many Fibonnaci primes.

5 Landau Open Problems with Remarks

At the 1912 International Congress of Mathematicians, Edmund Landau [7] raised the following four major open questions (conjectures) about prime numbers which were characterized in his speech at that time as "unattackable at the present state of science":
1. Goldbach Conjecture.
2. Twin Prime Conjecture.
3. Legendre Conjecture: For each natural number n, there is always a prime number p such that $n^2 < p < (n+1)^2$.
4. There are infinitely many prime numbers of the form $m^2 + 1$, where m is an integer.

Almost a century later, these problems have remained defiant. Indeed, as we mentioned earlier the first one had been open since Goldbach raised it in 1742 in a letter to Euler and the second one is a 2300-year-old mystery now. The third one resisted all attempts even though it is in the same spirit as Bertrand Postulate (Theorem) and the following theorem due to Ingham [6]: For every large enough natural number n, there is a prime number p such that $n^3 < p < (n+1)^3$.

Moreover, in [4] M. El Bachraoui raised a question which we rephrase as:
Is it true that for all integers $n > 1$ and a fixed integer $k \leq n$ there exists a prime number in the interval $(kn, (k+1)n]$?

The case $k = 1$ is Bertrand Postulate. El Bachraoui [4] gave a positive answer for the case $k = 2$. Andy Loo [5] gave a positive answer for the case $k = 3$. A positive answer to the problem would imply a positive answer to Legendre Conjecture (by taking $k = n$).

Finally, concerning the fourth problem, it can be obviously be phrased as:
There are infinitely many prime numbers of the form $n^2 + 1$, where n is a natural number.

it is known that there are infinitely many prime numbers of the forms x^2+1, where x is a real number. On the other hand, in 1978, Hendrik Iwaniec showed that there are infinitely many values of m^2+1 that are either primes or a product of two primes.

Finally, I show a method which could lead to the proof of problem 4. Suppose, to get a contradiction, that there were only finitely many primes of that form:

$p_1 = m_1^2 + 1, m_1 \in \mathbb{Z}$

$p_2 = m_2^2 + 1, m_2 \in \mathbb{Z}$

...

$p_k = m_k^2 + 1, m_k \in \mathbb{Z}$.

Clearly $p_2, p_3, ...p_k \geq 5$. Now

$$m_1^2 m_2^2 ... m_k^2 + 1 = (p_1 - 1)(p_2 - 1)...(p_k - 1) + 1.$$

Thus it suffices to show that $p = (p_2 - 1)...(p_k - 1) + 1$ is a prime number different from each of $p_2, ..., p_k$. Suppose, to get a contradiction, that $p = p_j$ for some $j = 2, 3, ..., k$. Then

$$p_j - 1 = (p_2 - 1)(p_3 - 1)...(p_k - 1).$$

So

$$1 = (p_{j_2} - 1)(p_{j_3} - 1)...(p_{j_k} - 1)$$

which is impossible as the right-hand side is even. Consequently, a positive answer to problem 4 hinges on just proving that p is a prime number.

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