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Two Kinds of Frobenius Problems in $\mathbb{Z}[\sqrt{m}]$

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Abstract

If m is a positive integer, not a perfect square, there are two obvious kinds of Frobenius problems that can be posed in $\mathbb{Z}[\sqrt{m}]$. One kind "lives in" $\mathbb{N}[\sqrt{m}]$, and the other lives in $\mathbb{Z}[\sqrt{m}]^+ = \mathbb{Z}[\sqrt{m}] \cap (0, \infty)$. We solve all instances of the second type of problem and add a bit to what is known about the first type.

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1 Introduction

$\mathbb{Z}$ will denote the set of integers, $\mathbb{N}$ the set of non-negative integers. Given $a_1, \ldots, a_n \in \mathbb{N}\backslash\{0\}$, let $SG(a_1, \ldots, a_n) = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$; $SG(a_1, \ldots, a_n)$ is called the *monoid* or *semigroup* generated by $a_1, \ldots, a_n$. Sometimes the role of $\mathbb{N}$ as the reservoir of the *coefficients* $\lambda_1, \ldots, \lambda_n$ is mentioned, but not usually.

Clearly $SG(a_1, \ldots, a_n) \subseteq \mathbb{N}$. If $a_1, \ldots, a_n$ are relatively prime (i.e., have no common divisor other than ± 1) then $SG(a_1, \ldots, a_n)$ contains a *tail* of $\mathbb{N}$, $\{g, g+1, \cdots\} = g+\mathbb{N}$, and the *Frobenius problem* associated with $a_1, \ldots, a_n$ is to find the smallest $g = g(a_1, \ldots, a_n)$ such that $g + \mathbb{N} \subseteq SG(a_1, \ldots, a_n)$. For instance, it is easy to see that $SG(3, 5) = \{0, 3, 5, 6, 8, 9, \ldots\}$, so $g(3, 5) = 8$.

These are called Frobenius problems because the great German mathematician Frobenius gave a memorable proof of a formula solution when $n = 2 : g(a_1, a_2) = (a_1 - 1)(a_2 - 1)$. (Others, including Sylvester [4], gave proofs of this result; Frobenius's proof was particularly nice, and proved more than just the formula.) For $n \geq 3$, things are not so simple, and Frobenius problems in $\mathbb{Z}$ is still an area of research with many opportunities for progress.

To understand the transplantation of Frobenius problems to rings other than the integers, as in [1] and [2], observe that when $a_1, \ldots, a_n \in \mathbb{N}\backslash\{0\}$ are coprime (relatively prime), then that tail $\{g, g+1, \ldots\} \subseteq SG(a_1, \ldots, a_n)$ that we aim to find can be described thus: $\{w \in \mathbb{Z} \mid w+\mathbb{N} \subseteq SG(a_1, \ldots, a_n)\}$, or $\{w \in SG(a_1, \ldots, a_n) \mid w + \mathbb{N} \subseteq SG(a_1, \ldots, a_n)\}$ or $\{w \in \mathbb{N} \mid w + \mathbb{N} \subseteq SG(a_1, \ldots, a_n)\}$. Being a subset of $\mathbb{N}$, this set has a least element, $g(a_1, \ldots, a_n)$, and the whole set is known once $g(a_1, \ldots, a_n)$ is known—but life may be more complicated in other rings.

Let m be a positive integer, not a perfect square, and $\mathbb{Z}[\sqrt{m}] = \{a+b\sqrt{m} \mid a, b \in \mathbb{Z}\}$. Because $\sqrt{m}$ is irrational, each $\alpha \in \mathbb{Z}[\sqrt{m}]$ is uniquely represented in the form $\alpha = a+b\sqrt{m}$, a, $b \in \mathbb{Z}$; we will sometimes call a the *rational part* or *rational component* of α, and b the irrational part or *irrational component* of α. Because we are not requiring m to be square-free, this naming of the irrational part could conflict with convention, and cause confusion. For instance, $2 + 15\sqrt{2} = 2 + 5\sqrt{18}$ has in our terminology, irrational part 15 in $\mathbb{Z}[\sqrt{2}]$ and irrational part 5 in $\mathbb{Z}[\sqrt{18}]$.

Another source of misunderstanding: if m is square-free, many algebraists take $\mathbb{Z}[\sqrt{m}]$ to stand for the intersection of $\mathbb{Q}[\sqrt{m}]$ (where $\mathbb{Q} = \{\text{rationals}\}$) with the ring of algebraic integers. For instance, with this interpretation, $\frac{1+\sqrt{5}}{2} \in \mathbb{Z}[\sqrt{5}]$ because $\frac{1+\sqrt{5}}{2}$ is a root of $x^2 - x - 1$, and is therefore an algebraic integer in $\mathbb{Q}[\sqrt{5}]$. While there are doubtless interesting Frobenius

problems in these rings, the "integral quadratic extensions" of $\mathbb{Z}$, and we hope that readers of this paper will be inspired to look into the matter, we will be sticking with our definition of $\mathbb{Z}[\sqrt{m}]$, according to which $\frac{1+\sqrt{5}}{2}$ is *not* an element of $\mathbb{Z}[\sqrt{5}]$.

With m as described (from now on), let $\mathbb{N}[\sqrt{m}] = \{a + b\sqrt{m} \mid a, b \in \mathbb{N}\}$ and $\mathbb{Z}[\sqrt{m}]^+ = \mathbb{Z}[\sqrt{m}] \cap [0, \infty) = \{\alpha \in \mathbb{Z}[\sqrt{m}] \mid \alpha \geq 0\}$. Clearly $\mathbb{N}[\sqrt{m}] \subseteq \mathbb{Z}[\sqrt{m}]^+$, and both $\mathbb{N}[\sqrt{m}]$ and $\mathbb{Z}[\sqrt{m}]^+$ are closed under addition and multiplication. For $\alpha_1, \ldots, \alpha_n \in \mathbb{N}[\sqrt{m}]\backslash\{0\}$, let $SG_0(\alpha_1, \ldots, \alpha_n) = \{\lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{N}[\sqrt{m}]\}$, and for $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[\sqrt{m}]^+\backslash\{0\}$, let $SG_1(\alpha_1, \ldots, \alpha_n) = \{\lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{Z}[\sqrt{m}]^+\}$. Clearly $SG_0(\alpha_1, \ldots, \alpha_n) \subseteq \mathbb{N}[\sqrt{m}]$ if $\alpha_1, \ldots, \alpha_n \in \mathbb{N}[\sqrt{m}]\backslash\{0\}$ and $SG_1(\alpha_1, \ldots, \alpha_n) \subseteq \mathbb{Z}[\sqrt{m}]^+$ if $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[\sqrt{m}]^+$. In the first case, SG_0 is closed under multiplication by elements of $\mathbb{N}[\sqrt{m}]$, and, in the second, SG_1 is closed under multiplication by elements of $\mathbb{Z}[\sqrt{m}]^+$. In each case, SG_i is closed under addition.

2 The two kinds of Frobenius problems

For $\alpha_1, \ldots, \alpha_n \in \mathbb{N}[\sqrt{m}]\backslash\{0\}$, the problem is to describe the set
$\text{Frob}_0(\alpha_1, \ldots, \alpha_n) = \{w \in \mathbb{Z}[\sqrt{m}] \mid w + \mathbb{N}[\sqrt{m}] \subseteq SG_0(\alpha_1, \ldots, \alpha_n)\}$.
The other problem type: for $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[\sqrt{m}]^+\backslash 0$, describe the set
$\text{Frob}_1(\alpha_1, \ldots, \alpha_n) = \{w \in \mathbb{Z}[\sqrt{m}] \mid w + \mathbb{Z}[m]^+ \subseteq SG_1(\alpha_1, \ldots, \alpha_n)\}$

It is elementary that, for $i = 0, 1, \text{Frob}_i(\alpha_1, \ldots, \alpha_n) \subseteq SG_i(\alpha_1, \ldots, \alpha_n)$, and if $\alpha_1, \ldots, \alpha_n$ are not coprime, i.e., if the α_i have some non-unit common divisor, then $\text{Frob}_i(\alpha_1, \ldots, \alpha_n) = \emptyset$. (This is also a consequence of Proposition 1, below.) Also: for any $w \in \mathbb{Z}[\sqrt{m}]$, if $\tilde{w} \in w + \mathbb{N}[\sqrt{m}]$ then $\tilde{w} + \mathbb{N}[\sqrt{m}] \subseteq w + \mathbb{N}[\sqrt{m}]$, and if $\tilde{w} \in w + \mathbb{Z}[\sqrt{m}]^+$, then $\tilde{w} + \mathbb{Z}[\sqrt{m}]^+ \subseteq w + \mathbb{Z}[\sqrt{m}]$. It follows that if $w \in \text{Frob}_0(\alpha_1, \ldots, \alpha_n)$ then $w + \mathbb{N}[\sqrt{m}] \subseteq \text{Frob}_0(\alpha_1, \ldots, \alpha_n)$, and if $w \in \text{Frob}_1(\alpha_1, \ldots, \alpha_n)$, then $w + \mathbb{Z}[\sqrt{m}]^+ \subseteq \text{Frob}_1(\alpha_1, \ldots, \alpha_n)$.

By analogy with Frobenius problems in $\mathbb{Z}$, we might hope that when $\text{Frob}_0(\alpha_1, \ldots, \alpha_n) \neq \emptyset$, perhaps $\text{Frob}_0(\alpha_1, \ldots, \alpha_n) = w + \mathbb{N}[\sqrt{m}]$ for some single $w \in SG_0(\alpha_1, \ldots, \alpha_n)$, and that when $\text{Frob}_1(\alpha_1, \ldots, \alpha_n) \neq \emptyset$, perhaps $\text{Frob}_1(\alpha_1, \ldots, \alpha_n) = w + \mathbb{Z}[\sqrt{m}]^+$ for some single $w \in SG_1(\alpha_1, \ldots, \alpha_n)$. The main result of section 3 is that this latter wish comes true, with the same w every time: if $\text{Frob}_1(\alpha_1, \ldots, \alpha_n) \neq \emptyset$, then $\text{Frob}_1(\alpha_1, \ldots, \alpha_n) = \mathbb{Z}[\sqrt{m}]^+ = 0 + \mathbb{Z}[\sqrt{m}]^+$. In the case of $\text{Frob}_0(\alpha_1, \ldots, \alpha_n)$, it can be proven that if this set is non-empty then it is a union of a finite number of sets of the form $w + \mathbb{N}[\sqrt{m}]$. We omit the proof, but it starts: first, consider all elements

of $\mathrm{Frob}_0(\alpha_1, \ldots, \alpha_n)$ with smallest irrational part .... We have not yet run across a case in which $\mathrm{Frob}_0(\alpha_1, \ldots, \alpha_n) \neq \emptyset$ and $\mathrm{Frob}_0(\alpha_1, \ldots, \alpha_m)$ is not of the form $w + \mathbb{N}[\sqrt{m}]$, but we have not proven that $\mathrm{Frob}_0(\alpha_1, \ldots, \alpha_n) \neq \emptyset$ will always be of this form—there's an open problem! See section 4.

We will say that a sequence $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[\sqrt{m}]$ *spans unity in* $\mathbb{Z}[\sqrt{m}]$ if, for some $\beta_1, \ldots, \beta_n \in \mathbb{Z}[\sqrt{m}]$, $1 = \alpha_1\beta_1 + \cdots + \alpha_n\beta_n$. In other words $\alpha_1, \ldots, \alpha_n$ spans unity in $\mathbb{Z}[\sqrt{m}]$ if the ideal generated by $\alpha_1, \ldots, \alpha_n$ in $\mathbb{Z}[\sqrt{m}]$ is all of $\mathbb{Z}[\sqrt{m}]$.

Proposition 1. *If either* $\alpha_1, \ldots, \alpha_n \in \mathbb{N}[\sqrt{m}]\backslash\{0\}$ *and* $Frob_0(\alpha_1, \ldots, \alpha_n) \neq \emptyset$, *or* $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[\sqrt{m}]^+\backslash\{0\}$ *and* $Frob_1(\alpha_1, \ldots, \alpha_n) \neq \emptyset$, *then the sequence* $\alpha_1, \ldots, \alpha_n$ *spans unity in* $\mathbb{Z}[\sqrt{m}]$.

Proof. In either case, take $w \in \mathrm{Frob}_i(\alpha_1, \ldots, \alpha_n)$. Then both w and $w + 1$ are in $SG_i(\alpha_1, \ldots, \alpha_n)$, so $1 = (w+1) - w = \sum_{i=1}^{n} \beta_i\alpha_i$ for some $\beta_1, \ldots, \beta_n \in \mathbb{Z}[\sqrt{m}]$. □

From [2] we have the following on the non-triviality of $\mathrm{Frob}_i(\alpha_1, \ldots, \alpha_n)$.

Theorem 1. *Suppose that the sequence* $\alpha_1, \ldots, \alpha_n \in \mathbb{N}[\sqrt{m}]\backslash\{0\}$ *spans unity in* $\mathbb{Z}[\sqrt{m}]$. *Then* $Frob_0(\alpha_1, \ldots, \alpha_n) \neq \emptyset$ *if and only if at least one* α_i *has either rational or irrational part* 0.
Suppose that the sequence $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[\sqrt{m}]^+\backslash\{0\}$ *spans unity in* $\mathbb{Z}[\sqrt{m}]$. *Then* $Frob(\alpha_1, \ldots, \alpha_n) \neq \emptyset$.

Corollary 1. *If* $\alpha_1, \ldots, \alpha_n \in \mathbb{N}[\sqrt{m}\backslash\{0\}$, *and at least one* α_i *has either rational or irrational part* 0, *then* $Frob_0(\alpha_1, \ldots, \alpha_n) \neq \emptyset$ *if and only if the sequence* $\alpha_1, \ldots, \alpha_n$ *spans unity in* $\mathbb{Z}[\sqrt{m}]$. *If* $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[\sqrt{m}]^+\backslash\{0\}$, *then* $Frob_1(\alpha_1, \ldots, \alpha_n) \neq \emptyset$ *if and only if the sequence spans unity in* $\mathbb{Z}[\sqrt{m}]$.

The Corollary claims follow directly from the proposition and theorem preceding.

The two kinds of Frobenius problems in $\mathbb{Z}[\sqrt{m}]$ that we are considering are therefore not perfectly analogous to Frobenius problems in $\mathbb{Z}$: the hypothesis of coprimality has been replaced by the stronger hypothesis of spanning unity. These are equivalent in $\mathbb{Z}$, and also in $\mathbb{Z}[\sqrt{m}]$ for some values of m. For instance, if $\mathbb{Z}[\sqrt{m}]$ is a Euclidean ring, then the two hypotheses are equivalent. We wonder if it is known precisely for which values of m, a non-square positive integer, coprimality of $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[\sqrt{m}]$ implies that the sequence spans unity.

Here is a convenient sufficient condition for $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[\sqrt{m}]$ to span unity. First, for $\alpha = a + b\sqrt{m}$, $a, b \in \mathbb{Z}$, the *conjugate* of α in $\mathbb{Z}[\sqrt{m}]$

is $\bar{\alpha} = a - b\sqrt{m}$, and the *norm* of α is $N(\alpha) = |\alpha\bar{\alpha}| = |a^2 - mb^2|$. It is straightforward to see that conjugacy in $\mathbb{Z}[\sqrt{m}]$ is multiplicative: if α, $\beta \in \mathbb{Z}[\sqrt{m}]$, $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$. Therefore, N is multiplicative: $N(\alpha\beta) = N(\alpha)N(\beta)$. From this and the fact that $N(\alpha) = \pm\alpha\bar{\alpha}$, it is easy to see that $\alpha \in \mathbb{Z}[\sqrt{m}]$ has a multiplicative inverse if and only if $N(\alpha) = 1$. All of this is standard: see [3].

Proposition 2. *Suppose* $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[\sqrt{m}]$ *and* $N(\alpha_1), \ldots, N(\alpha_n)$ *are coprime in* $\mathbb{Z}$. *Then the sequence* $\alpha_1, \ldots, \alpha_n$ *spans unity in* $\mathbb{Z}[\sqrt{m}]$.

Proof. Because $N(\alpha_1), \ldots, N(\alpha_n)$ are coprime in $\mathbb{Z}$, there exist $z_1, \ldots, z_n \in \mathbb{Z}$ such that $1 = \sum_{i=1}^{n} z_i N(\alpha_i) = \sum_{i=1}^{n} \lambda_i \alpha_i$, $\lambda_i = \pm z_i \overline{\alpha_i} \in \mathbb{Z}[\sqrt{m}]$, $i = 1, \ldots, n$. □

3 $\mathbb{Z}[\sqrt{m}]^+$

Lemma 1. *If* $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[\sqrt{m}]^+ \backslash \{0\}$, *then* $Frob_1(\alpha_1, \ldots, \alpha_n) = \mathbb{Z}[\sqrt{m}]^+$ *if and only if* $1 \in SG_1(\alpha_1, \ldots, \alpha_n)$.

Proof. If $\mathrm{Frob}_1(\alpha_1, \ldots, \alpha_n) = \mathbb{Z}[m]^+$, then

$$1 = 1 + 0\sqrt{m} \in \mathbb{Z}[\sqrt{m}]^+ \quad = \mathrm{Frob}_1(\alpha_1, \ldots, \alpha_n) \subseteq SG_1(\alpha_1, \ldots, \alpha_n).$$

On the other hand, if $1 = \lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}[\sqrt{m}]^+$, then for any $\beta \in \mathbb{Z}[\sqrt{m}]^+$,

$$\beta = (\beta\lambda_1)\alpha_1 + \cdots + (\beta\lambda n)\alpha_n \in SG(\alpha_1, \ldots, \alpha_n).$$

Therefore $\mathbb{Z}[\sqrt{m}]^+ \subseteq SG(\alpha_1, \ldots, \alpha_n) \subseteq \mathbb{Z}[\sqrt{m}]^+$. Therefore $0 + \mathbb{Z}[\sqrt{m}]^+ \subseteq \mathrm{Frob}_1(\alpha_1, \ldots, \alpha_n) \subseteq \mathbb{Z}[\sqrt{m}]^+$. □

Lemma 2. $\mathbb{Z}[\sqrt{m}]$ *is dense in the set of real numbers.*

Proof. It is well know that if δ is an irrational real number, then $\{a + b\delta \mid a, b \in \mathbb{Z}\}$ is dense in the reals. The conclusion follows from the previously noted fact that $\sqrt{m}$ is irrational. □

Theorem 2. *If the sequence* $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[\sqrt{m}]^+ \backslash \{0\}$ *spans unity in* $\mathbb{Z}[\sqrt{m}]$, *then* $Frob_1(\alpha_1, \ldots, \alpha_n) = \mathbb{Z}[\sqrt{m}]^+$.

Proof. By Lemma 1 it suffices to show that $1 = \mu_1\alpha_1 + \cdots + \mu_n\alpha_n$ for some $\mu_1, \ldots, \mu_n \in \mathbb{Z}[\sqrt{m}]^+$. By hypothesis, $1 = \lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}[\sqrt{m}]$. For $i = 1, \ldots, n$, let $\beta_i = (\prod_{j=1}^n \alpha_j)/\alpha_i \in \mathbb{Z}[\sqrt{m}]^+$.

For any $\gamma_1, \ldots, \gamma_n \in \mathbb{Z}[\sqrt{m}]$ such that $\sum_{i=1}^n \gamma_i = 0$, if $\mu_i = \lambda_i + \gamma_i\beta_i$, $i = 1, \ldots, n$, then $1 = \sum_{i=1}^n \mu_i\alpha_i$. If $n = 1$ then α_1 is a positive unit in $\mathbb{Z}[\sqrt{m}]$; therefore $\lambda_1 = \alpha_1^{-1} > 0$ and we are done.

So suppose that $n \geq 2$. By the density of $\mathbb{Z}[\sqrt{m}]$ in $\mathbb{R}$ (Lemma 2) we can find $\gamma_i \in \mathbb{Z}[\sqrt{m}] \cap (-\frac{\lambda_i}{\beta_i}, -\frac{\lambda_i}{\beta_i} + \frac{1}{n\alpha_i\beta_i})$, $i = 1, \ldots, n-1$. If we set $\gamma_n = -\sum_{i=1}^{n-1} \gamma_i$ we then have $\sum_{i=1}^n \gamma_i = 0$, so, if $\mu_i = \lambda_i + \gamma_i\beta_i$, then $1 = \sum_{i=1}^n \mu_i\alpha_i$. If $i \in \{1, \ldots, n-1\}$, $0 < \mu_i < \frac{1}{n\alpha_i}$; therefore $1 = \sum_{i=1}^n \mu_i\alpha_i < \frac{n-1}{n} + \mu_n\alpha_n$, so $\mu_1, \ldots, \mu_n > 0$. Thus $\mu_1, \ldots, \mu_n \in \mathbb{Z}[\sqrt{m}]^+$. □

Corollary 2. *If $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[\sqrt{m}]^+\backslash\{0\}$, then $Frob_1(\alpha_1, \ldots, \alpha_n) = \mathbb{Z}[\sqrt{m}]^+$ if and only if $Frob_1(\alpha_1, \ldots, \alpha_n) \neq \emptyset$.*

Proof. The conclusion follows from the preceding theorem and Proposition 1. □

Corollary 3. *If $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[\sqrt{m}]^+\backslash\{0\}$ and $N(\alpha_1), \ldots, N(\alpha_n)$ are coprime in $\mathbb{Z}$, then*

$$Frob_1(\alpha_1, \ldots, \alpha_m) = \mathbb{Z}[\sqrt{m}]^+.$$

The corollary follows from Theorem 2 and Proposition 2.

4 $N[\sqrt{m}]$

The problem of describing $\mathrm{Frob}_0(\alpha_1, \ldots, \alpha_n)$ for $\alpha_1, \ldots \alpha_n \in \mathbb{N}[\sqrt{m}]\backslash\{0\}$ for $\alpha_1, \ldots, \alpha_n$ spanning unity, with at least one α_i purely rational or purely irrational, will surely be no easier than solving a Frobenius problem, in $\mathbb{Z}$, given n coprime elements of $\mathbb{N}$. For coprime $a_1, \ldots, a_n \in \mathbb{N}\backslash\{0\}$, let $g(a_1, \ldots, a_n)$ be the solution of the Frobenius problem in $\mathbb{Z}$ associated with $a_1, \ldots, a_n$, as discussed in the Introduction. We will knock off here two of the easy cases of determining $\mathrm{Frob}_0(\alpha_1, \ldots, \alpha_n)$ in the cases when it is non-empty.

Theorem 3. *Suppose that $a_1, \ldots, a_n \in \mathbb{Z}\backslash\{0\}$ are coprime in $\mathbb{Z}$. Regarding $a_1, \ldots, a_n$ as elements of $\mathbb{N}[\sqrt{m}]\backslash\{0\}$,*

$$Frob_0(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)(1 + \sqrt{m}) + \mathbb{N}[\sqrt{m}]$$

Proof. The proof is straightforward from the observation that if $c_i, d_i \in \mathbb{N}$, $i = 1, \ldots, n$, then $\sum_{i=1}^n a_i(c_i + d_i\sqrt{m}) = \sum a_ic_i + (\sum a_id_i)\sqrt{m}$. □

Lemma 3. *Suppose that* $a_1, \ldots, a_r, b_1, \ldots, b_s \in \mathbb{Z}$.
The sequence $a_1, \ldots, a_r, b_1\sqrt{m}, \ldots, b_s\sqrt{m}$ *spans unity in* $\mathbb{Z}[\sqrt{m}]$ *if and only if* $a_1, \ldots, a_r, b_1 m, \ldots, b_s m$ *are coprime in* $\mathbb{Z}$.

Proof. If $a_1, \ldots, a_r$, $b_1 m, \ldots, b_s m$ are coprime in $\mathbb{Z}$, then they span unity in $\mathbb{Z}$; that is, for some $c_1, \ldots, c_r$, $d_1, \ldots, d_s \in \mathbb{Z}$,

$$\begin{aligned} 1 &= \textstyle\sum_{i=1}^{r} c_i a_i + \sum_{j=1}^{s} d_j b_j m \\ &= \textstyle\sum_{i=1}^{r} c_i a_i + \sum_{j=1}^{s} (d_j\sqrt{m}) b_j \sqrt{m}. \end{aligned}$$

Therefore, $a_1, \ldots, a_r$, $b_1\sqrt{m}, \ldots, b_s\sqrt{m}$ span unity in $\mathbb{Z}[\sqrt{m}]$.

If $a_1, \ldots, a_r$, $b_1\sqrt{m}, \ldots, b_s\sqrt{m}$ span unity in $\mathbb{Z}[\sqrt{m}]$, then for some $c_i, d_i, e_j, f_j \in \mathbb{Z}$, $i = 1, \ldots, r, j = 1, \ldots, s$,

$$\begin{aligned} 1 &= \textstyle\sum_{i=1}^{r} a_i(c_i + d_i\sqrt{m}) + \sum_{j=1}^{s} b_j\sqrt{m}(e_j + f_j\sqrt{m}) \\ &= \textstyle\sum_{i=1}^{r} a_i c_i + \sum_{j=1}^{s} f_j(b_j m) + (\sum_{i=1}^{r} a_i d_i + \sum_{j=1}^{s} b_j e_j)\sqrt{m} \end{aligned}$$

Therefore, $1 = \sum_{i=1}^{4} a_i c_i + \sum_{j=1}^{s} f_j(b_j m)$, so $a_1, \ldots, a_r, b_1 m, \ldots, b_s m$ must be coprime in $\mathbb{Z}$. □

Theorem 4. *Suppose that* $a_1, \ldots, a_r, b_1, \ldots, b_s$ *are positive integers, and that* $a_1, \ldots, a_r, b_1 m, \ldots, b_s m$ *are coprime in* $\mathbb{Z}$. *Then*

$$\begin{aligned} &\mathit{Frob}_0(a_1, \ldots, a_r, b\sqrt{m}, \ldots, b_s\sqrt{m}) \\ &= g(a_1, \ldots, a_r, b_1 m, \ldots, b_s m) + g(a_1, \ldots, a_r, b_1, \ldots, b_s)\sqrt{m} + \mathbb{N}[\sqrt{m}] \end{aligned}$$

Proof. Note that the coprimality of $a_1, \ldots, a_r, b_1 m, \ldots, b_s m$ implies that of $a_1, \ldots, a_r, b_1, \ldots, b_s$.

Suppose $c_i, d_i, e_j, f_j \in \mathbb{N}$, $i = 1, \ldots, r$, $j = 1, \ldots, s$. We have

$$\begin{aligned} &\textstyle\sum_{i=1}^{r} a_i(c_i + d_i\sqrt{m}) + \sum_{j=1}^{s} (b_j\sqrt{m})(e_j + f_j\sqrt{m}) = \\ &\textstyle\sum_{i=1}^{r} a_i c_i + \sum_{j=1}^{s} f_j(b_j m) + (\sum_{i=1}^{r} a_i d_i + \sum_{j=1}^{s} b_j e_j)\sqrt{m}; \end{aligned}$$

freely choosing $c_i, d_i, e_j, f_j \in \mathbb{N}$ we can get every rational part from $g(a_1, \ldots, a_r, b_1 m, \ldots, b_s m)$ on in the result, but not $g(a_1, \ldots, a_r, b_1 m, \ldots, b_s m) - 1$, and, independently, every irrational part from $g(a_1, \ldots, a_r, b_1, \ldots, b_s)$ onward, but never one less than this number. □

Clearly Theorem 3 can be viewed as a special case of Theorem 4, but we thought it best to sacrifice elegance for clarity.

In the cases remaining, $\alpha_1, \ldots, \alpha_n \in \mathbb{N}[\sqrt{m}]\backslash\{0\}$, $\alpha_1, \ldots, \alpha_n$ span unity in $\mathbb{Z}[\sqrt{m}]$, at least one α_i has either rational or irrational part 0, and some other α_j has both parts positive. We leave these cases open. Obviously, it will be best to start with $n = 2$.

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