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On the solution of integral equations of the first kind with singular kernels of Cauchy-type

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Abstract

Two efficient quadrature formulae have been developed for evaluating numerically certain singular integral equations of the first kind over the finite interval [-1,1]. Central to this work is the application of four special cases of the Jacobi polynomials $\mathrm{P}_n^{\alpha,\beta}(x)$, whose zeros served as interpolation and collocation nodes: (i) $\alpha = \beta = -\frac{1}{2}$, $\mathrm{T}_n(x)$, the first kind Chebyshev polynomials (ii) $\alpha = \beta = \frac{1}{2}, \mathrm{U}_n(x)$, the second kind (iii) $\alpha = -\frac{1}{2}, \beta = \frac{1}{2}, \mathrm{V}_n(x)$, the third kind (iv) $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}, \mathrm{W}_n(x)$, the fourth kind. Four tables of numerical results have been provided for verification and validation of the rules developed.

1 Introduction

Cauchy singular integral equations of the first kind are generally expressed as

$$\frac{1}{\pi} \int_{-1}^{1} \frac{k_1(x,s)\phi(x)}{x-s} dx + \frac{1}{\pi} \int_{-1}^{1} k_2(x,s)\phi(x) dx = g(s), \qquad -1 \leq s \leq 1 \quad (1.1)$$

where k_1, k_2 and g are real valued functions which satisfy the Hölder condition with respect to each of the independent variables and ϕ is the solution

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to be sought. These equations are encountered in aerodynamics and plane elasticity [5] and a variety of problems of mathematical physics [7], such as fracture problems in solid mechanics. Such integral equations arise quite generally from problems involving the scattering of radiation. Many authors, including Mushkhelishvili [8], Ioakimidis [4], Tricomi [10], Srivastav [9], and a host of others have investigated this problem and have offered analytical insight and numerical solutions for various forms of (1). For instance, Kim[6] solved (1) using Gaussian rule with zeros of Chebyshev polynomials of the second kind as the nodes, while the zeros of the first kind were taken to be the collocation points. The most recent paper on this problem is by Eshkuvatov, Long, and Abdukawi [2] who approximated the unknown function ϕ by a finite series of orthogonal polynomials and then employed the usual collocation 'trick', and further showed analytically that the method would be exact whenever the output function $g(s)$ is linear.

In this paper we set $k_1 = 1$ and $k_2 = 0$, to obtain

$$\fint_{-1}^{1} \frac{\phi(x)}{x-s} dx = g(s) \qquad -1 \leq s \leq 1 \tag{1.2}$$

The principal concern of this work is the numerical approximation of (2). This equation has a practical problem; it has a singularity of Cauchy-type. According to Mushkhelishvili [8], the analytical solution of (1) falls into four categories:

Category(i): *Solution is unbounded at both end-points* $s = \pm 1$

$$\phi(s) = \frac{f(s)}{\sqrt{1-s^2}} \tag{1.3}$$

and for uniqueness of solution is imposed the condition

$$\int_{-1}^{1} \phi(x) dx = 0 \tag{1.4}$$

where $f(s)$ is a bounded function on [-1,1].

Category(ii): *Solution is bounded at both end-points* $s = \pm 1$

$$\phi(s) = h(s)\sqrt{1-s^2} \tag{1.5}$$

subject to

$$\int_{-1}^{1} \frac{g(s)}{\sqrt{1-s^2}} ds = 0 \tag{1.6}$$

where h(s) is a bounded function on [-1,1]
Category(iii): *Solution is bounded at one end-point* $s = -1$

$$\phi(s) = \sqrt{\frac{1+s}{1-s}}y(s) \tag{1.7}$$

where y(s) is a bounded function on [-1,1].
Category(iv): *Solution is bounded at one end-point* $s = 1$

$$\phi(s) = \sqrt{\frac{1-s}{1+s}}q(s) \tag{1.8}$$

where $q(s)$ is a bounded function on [-1,1]
The sequence, $\frac{1}{\sqrt{1-x^2}}$, $\sqrt{1-x^2}$, $\sqrt{\frac{1+x}{1-x}}$, $\sqrt{\frac{1-x}{1+x}}$ is the set of weight functions of the Chebyshev polynomials of the first, second, third and fourth kinds respectively. We denote these weight functions by the sequence $\{w_j(x)\}_{j=1}^{4}$ respectively and the corresponding orthogonal polynomials by $T_n(x), U_n(x)$, $V_n(x), W_n(x)$. These polynomials are special cases of the Jacobi polynomials $\mathrm{P}_n^{\alpha,\beta}(x)$ and appear in potential theory. All the four Chebyshev polynomials satisfy the same recurrence relation [see [3]]

$$z_{n+1} = 2tz_n - z_{n-1}, \quad n = 1, 2, ..., \tag{1.9}$$

where

$$z_0 = 1 \text{ and } z_1 = \begin{cases} t & \text{for } T_n(t) \\ 2t & \text{for } U_n(t) \\ 2t-1 & \text{for } V_n(t) \\ 2t+1 & \text{for } W_n(t) \end{cases}$$

This paper is outlined as follows: In Section 2 we give briefly some properties of the four Chebyshev polynomials. In Section 3 we outline our method which evolves from the use of Lagrange interpolation formula, Christoffel-Darboux identity, collocation technique and the polynomial properties of Section 2. To verify and validate our methods, we present in Section 4 a numerical experiment and its approximate results. All computations were performed in Matlab code and in 'format long' mode (15 decimal digits) .

2 Some useful properties of the polynomials

It is known (see [2]) that

$$\int_{-1}^{1}\frac{w_1(x)T_n(x)}{x-s}dx=\pi U_{n-1}(s),\qquad \int_{-1}^{1}\frac{w_2(x)U_n(x)}{x-s}dx=-\pi T_{n+1}(s)$$

$$\int_{-1}^{1}\frac{w_3(x)V_n(x)}{x-s}dx=\pi W_n(s),\qquad \int_{-1}^{1}\frac{w_4(x)W_n(x)dx}{x-s}dx=-\pi V_n(s)$$

$$\frac{1}{\pi}\int_{-1}^{1}w_j(t)dt=1,\quad j=1,...,4,\qquad T_0=U_0=V_0=W_0=1$$

and in [3], given that $\theta=\cos^{-1}(x)$,

$$T_n(\cos\theta)=\cos n\theta\ ;\qquad \text{zeros}\ \ x_{1k}=\cos\left((2k-1)\frac{\pi}{2n}\right),\ k=1,...,n$$

$$U_n(\cos\theta)=\frac{\sin(n+1)\theta}{\sin\theta}\ ;\qquad \text{zeros}\ \ x_{2k}=\cos\left(\frac{k\pi}{n+1}\right),\ k=1,...,n$$

$$V_n(\cos\theta)=\frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}\ ;\qquad \text{zeros}\ \ x_{3k}=\cos\left((2k-1)\frac{\pi}{2n+1}\right),\ k=1,...,n$$

$$W_n(\cos\theta)=\frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\ ;\qquad \text{zeros}\ \ x_{4k}=\cos\left(\frac{2k\pi}{2n+1}\right),\ k=1,...,n$$

3 Approximate solution method

Let $H_n(x)$ be any of the four polynomials, which implies that $H_n\in\mathcal{D}=\{T_n(x),U_n(x),V_n(x),W_n(x)\}$ and $w(x)$ its corresponding weight function so that $w(x)\in\{w_1(x),w_2(x),w_3(x),w_4(x)\}$.
Following Mushkhelishvili [8] findings on the analytical solutions of (1), one may take the unknown function ϕ as:

$$\phi(x)=w(x)h(x) \tag{3.10}$$

where $h:[-1,1]\to R$ is some bounded function in R,and $w(x)$ the given weight functions as previously defined.
Substituting (10) in (2) gives

$$\int_{-1}^{1}\frac{w(x)h(x)}{x-s}dx=g(s) \tag{3.11}$$

Let $x_1,x_2,...,x_n$ be the zeros of $H_n(x)$. Then by the Lagrange interpolation formula,

$$\text{h(x)} = \sum_{i=1}^{n} \frac{H_n(x)h(x_i)}{(x - x_i)H_n'(x_i)} + e_n(x) \tag{3.12}$$

Ignoring the error term e_n, we have

$$\sum_{i=1}^{n} \frac{h(x_i)}{H_n'(x_i)} \int_{-1}^{1} \frac{w(x)H_n(x)}{(x - x_i)(x - s)} dx = g(s) \tag{3.13}$$

Define:

$$\Psi_n(s) = \int_{-1}^{1} \frac{w(x)H_n(x)}{x - s} dx \tag{3.14}$$

and note that Ψ_n can be obtained analytically by virtue of the orthogonal polynomial properties in Section 2. Observe also that $\Psi(s)$, by virtue of (9), satisfies the recurrence relation
$\Psi_{n+1}(s) = 2s\Psi_n(s) - \Psi_{n-1}(s) + 2\lambda_n$.
where

$$\begin{aligned} \lambda_n &= 0, \quad \text{if } \; n \geq 1 \\ \lambda_0 &= \int_{-1}^{1} w(x)dx, \quad n = 0 \end{aligned}$$

By this definition and by methods of partial fractions, we have

$$\sum_{i=1}^{n} \frac{h(x_i)}{H_n'(x_i)(s - x_i)} [\Psi_n(s) - \Psi_n(x_i)] = g(s) \tag{3.15}$$

Let $s_j, \; j = 1, ..., n$, be the zeros of $\hat{H}_n(x)$; $H_n(x) \neq \hat{H}_n(x) \in \mathcal{D}$.
Collocating at these points leads to the following system of linear equations in which h is to be determined,

$$\sum_{i=1}^{n} \frac{h(x_i)}{H_n'(x_i)(s_j - x_i)} [\Psi_n(s_j) - \Psi_n(x_i)] = g(s_j), \quad j = 1, ..., n \tag{3.16}$$

and in matrix form, written as

$$A\mathbf{h} = \mathbf{b} \tag{3.17}$$

where

$$\begin{aligned} \mathrm{A} = (a_{j,i})_{j,i} &= \frac{[\Psi_n(s_j) - \Psi_n(x_i)]}{H_n'(x_i)(s_j - x_i)}, \quad \mathbf{b}^T = [g(s_1), ..., g(s_n)] \\ \mathbf{h}^T &= [\tilde{h}(x_1), ..., \tilde{h}(x_n)]. \end{aligned}$$

On obtaining h from (17), the approximate solution at x_i then is $\tilde{\phi} = w(x_i)\tilde{h}(x_i)$. If, however, solutions of **Category(i)** type are desired, then an additional equation will have to come from equation(4); in the case, the collocation knots may be chosen to satisfy $\hat{H}_{n-1}(s_j) = 0,\ j = 1, ..., n-1$ and one may choose $\hat{H}_{n-1}(x) = U_{n-1}(x)$. Then the additional equation required is

$$\sum_{i=1}^{n} \pi \frac{U_{n-1}(x_i)h(x_i)}{T_n'(x_i)} = 0 \tag{3.18}$$

which simplifies to

$$\sum_{i=1}^{n} h(x_i) = 0 \tag{3.19}$$

and the system of equations to solve then is

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \frac{h(x_i)}{H_n'(x_i)(s_j - x_i)}[\Psi(s_j) - \Psi(x_i)] = g(s_j), \quad j = 1, ...n-1 \\ \displaystyle\sum_{i=1}^{n} h(x_i) = 0 \end{cases} \qquad [*]$$

Next, we derive an additional approximate rule by using Christoffel-Darboux formula [1] on (13).

Given any orthogonal polynomials $\mathrm{P}_n(t)$ with the weight functions $w(t)$ on $[a, b]$, define $\rho_n = \int_a^b w(t)\mathrm{P}_n^2(t)dt$ and $\mathrm{P}_n(t) = k_n t^n + ..., k_0$.

k_n is the leading coefficient of the polynomial P_n, degree n or the coefficient of the term t^n in $P_n(t)$; ρ_n is the inner product $< P_n, P_n >_{w(t)}$ with respect to the weight function $w(t)$ over $[a, b]$.

For most classical orthogonal polynomials the pairs (k_n, ρ_n) are very well known quantitatively, see for example [1].

Applying Christoffel-Darboux formula to (13) leads to another approximate formula

$$\sum_{i=1}^{n} \frac{h(x_i)\rho_n k_{n+1}}{H_n'(x_i)k_n H_{n+1}(x_i)} \sum_{m=0}^{n-1} \frac{H_m(x_i)\Psi_m(s)}{\rho_m} = -g(s) \qquad (3.20)$$

Collocating as in the previous case leads to the $n \times n$ system of linear equations in h.

$$\sum_{i=1}^{n} \left[\frac{k_{n+1}\rho_n}{k_n H_n'(x_i) H_{n+1}(x_i)} \sum_{m=0}^{n-1} \frac{H_m(x_i)}{\rho_m} \Psi_m(s_j) \right] h(x_i) = -g(s_j), \ j = 1, ..., n \qquad (3.21)$$

Again, should you want solutions of **Category(i)** type, then combine equations (21) and (19) with $j = 1, ...n$ in (21) changed to $j = 1, ..., n-1$ and the approximate solution obtained as in the preceding case. For exposition we put them together as

$$\begin{cases} \sum_{i=1}^{n} \left[\frac{k_{n+1}\rho_n}{k_n H_n'(x_i) H_{n+1}(x_i)} \sum_{m=0}^{n-1} \frac{H_m(x_i)}{\rho_m} \Psi_m(s_j) \right] h(x_i) = -g(s_j) \\ \qquad\qquad j = 1, ..., n-1 \qquad [**] \\ \sum_{i=1}^{n} h(x_i) = 0 \end{cases}$$

The linear systems of equations were all solved using gaussian elimination method with scaled partial pivoting in Matlab code.
Observe that both rules have the same numbers of functional evaluations and share the same degree of precision but differ in the number of multiplications necessary to implement them: the first rule, $2n^2$ and the latter, $6n^3$. Nonetheless, the latter has an edge over the former for being independent of the factor $|s_j - x_i|$ which might probably effect a 'cancellation effect' if it gets quite small for some i, j.

4 Proof of convergence

Lemma:
Given any function $f(x)$ of bounded variation in $[a, b]$ there can be found a polynomial $Q_n(x)$, degree n, such that

$|f(x) - Q_n(x)| < \epsilon$, whenever $n \to \infty,\ \epsilon \to 0$, (Jackson's theorem)

Theorem 4.1. *Let h_n be the Lagrange approximating polynomial interpolating to h at a finite number of chosen knots in $[a, b]$. Then the approximate rules of Section 3 converge and the error bound given by*
$|E_n| \leq \epsilon.\kappa$

Proof :
In view of (11) and (12), the error E_n can be expressed as,

$$E_n = \int_{-1}^{1} w(x) \frac{(h(x) - h(x)_n)}{x - s} dx = g - g_n.$$

By the preceding lemma

$$|E_n| \leq \epsilon \int_{-1}^{1} \frac{w(x)}{x - s} dx$$

and for the weight functions $w(x)$ under discussion this integral can be obtained in a closed form very readily; assuming the integral to be the constant $\kappa > 0$,
$|E_n| \leq \epsilon.\kappa$

5 Numerical Experiments And Results

Consider the following exceedingly simple structured singular integral equation for numerical experiment, verification, and validation of the approximate rules developed. We have deliberately taken the following problem from [2] for the sole purpose of comparing our distinct numerical approaches in accuracy and efficiency.

$$\int_{-1}^{1} \frac{\phi(x)}{x - s} dx = 4s^3 + 2s^2 + 3s - 1, \qquad -1 < s < 1 \tag{5.22}$$

The analytical solution, obtained by some results of [2], is given under four circumstances.
case(i): $w(x) = \dfrac{1}{\sqrt{1 - x^2}}$; solution unbounded at both end-points $s = \pm 1$

$$\phi(s) = \frac{1}{\pi\sqrt{1 - s^2}} \left(4s^4 + 2s^3 + s^2 - 2s - 2\right) \tag{5.23}$$

case(ii): $w(x) = \sqrt{1-x^2}$; solution is bounded at both end-points $s = \pm 1$

$$\phi(s) = -\frac{1}{\pi}\sqrt{1-s^2}\ (4s^2 + 2s + 5) \tag{5.24}$$

case(iii): $w(x) = \sqrt{\dfrac{1+x}{1-x}}$; solution bounded at one end-point $s = -1$.

$$\phi(s) = \frac{1}{\pi}\sqrt{\frac{1+s}{1-s}}\ (4s^3 - 2s^2 + 3s - 5) \tag{5.25}$$

case(iv): $w(x) = \sqrt{\dfrac{1-s}{1+s}}$; solution bounded at one end-point $x = 1$

$$\phi(s) = -\frac{1}{\pi}\sqrt{\frac{1-s}{1+s}}\ (4s^3 + 6s^2 + 7s + 5) \tag{5.26}$$

The numerical results that are to follow are for two cases only, namely case(i) and case(iii). For case(i) we had applied the rules [*] and [**] to (22) and the results are respectively depicted in Tables I and II. And for case(iii) we had used the rules (16) and (21) to solve (22) and the results are depicted in Tables III and IV. We did not find it necessary to repeat the computation process for the remaining two cases to avoid what might look like a repetition.

n=6 : case(i) using rule[*]

x_i	Approx($\tilde{\phi}$)	Exact(ϕ)	Abs(Error)
.966	2.811022891072834	2.811022891072835	0.000000000000000
.707	-0.543388965223067	-0.543388965223067	0.000000000000000
.259	-0.790242373555593	-0.790242373555593	0
-.259	-0.471932487371802	-0.471932487371803	0.000000000000001
-.707	0.093230807144514	0.093230807144514	0.000000000000000
-.966	3.129332777256615	3.129332777256617	0.000000000000002

Table I

n=6 : case(i) using rule [**]

x_i	Approx($\tilde{\phi}$)	Exact(ϕ)	Abs(Error)
.966	2.811022891072831	2.811022891072835	0.000000000000003
.707	-0.543388965223067	-0.543388965223067	0.000000000000000
.259	-0.790242373555593	-0.790242373555593	0.000000000000000
-.259	-0.471932487371802	-0.471932487371803	0.000000000000000
-.707	0.093230807144513	0.093230807144514	0.000000000000001
-.966	3.129332777256622	3.129332777256617	0.000000000000006

Table II

What follows next is the result of the numerical experiment carried on (22) using respectively rules (16) and (21) with the zeros of $V_n(x)$ as the interpolation nodes and the zeros of $W_n(x)$ as the collocation knots.

n=6 : case(iii) using rule (16)

x_i	Approx($\tilde{\phi}$)	Exact(ϕ)	Abs(Error)
0.970	-0.816063729038973	-0.816063729038968	0.000000000000005
0.748	-1.844423936634620	-1.844423936634618	0.000000000000002
0.354	-1.848902000605444	-1.848902000605438	0.000000000000006
-0.120	-1.522132836716184	-1.522132836716183	0.000000000000001
-0.568	-1.350335032174419	-1.350335032174416	0.000000000000004
-0.885	-0.941581028045417	-0.941581028045418	0.000000000000002

Table III

n=6 : case(iii) using rule (21)

x_i	Approx($\tilde{\phi}$)	Exact(ϕ)	Abs(Error)
0.970	-0.816063729038963	-0.816063729038968	0.000000000000005
0.748	-1.844423936634619	-1.844423936634618	0.000000000000001
0.354	-1.848902000605436	-1.848902000605438	0.000000000000001
-0.120	-1.522132836716183	-1.522132836716183	0.000000000000000
-0.568	-1.350335032174416	-1.350335032174416	0.000000000000000
-0.885	-0.941581028045424	-0.941581028045419	0.000000000000005

Table IV

6 Conclusion

Two quadrature formulae for evaluating singular integral equations of the first kind and in the form $\int_{-1}^{1} \frac{\phi(x)}{x-s}dx = g(s)$ have been developed. The outcomes of the numerical experiment indicate that the formulae are excellent and competitive. Whereas the paper in [2] needed $n = 20$ to achieve an accuracy of 10^{-16}, we only needed $n = 6$ to achieve the same accuracy.

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