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An Integral Inequality for Probability Spaces

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Abstract

An inequality for double integrals involving two measurable non-negative functions on a probability space is proved, using a special case of Hölder's inequality.

Throughout, (X, μ) will be a probability space.

Lemma 1. *Suppose that* $1 \leq r < \infty$ *and* $f \in L^r(\mu)$ *is non-negative. Then* $\int_X f^r d\mu \geq (\int_X f d\mu)^r$. *If* $r > 1$, *then equality holds if and only if* f *is (essentially) constant.*

Proof. The inequality holds with equality when $r = 1$. Suppose that $r > 1$. By Hölder's inequality,

$$\begin{aligned} \int_X f d\mu &\leq \left(\int_X f^r d\mu\right)^{1/r} \left(\int_X 1^{\frac{r}{r-1}} d\mu\right)^{1-\frac{1}{r}} \\ &= \left(\int_X f^r d\mu\right)^{1/r}, \end{aligned}$$

because $\mu(X) = 1$. Raising both sides of this equality to the power gives the result. The condition for each equality comes from the corresponding condition in Hölder's inequality. □

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Convention: $0^0 = 1$

Theorem 1. *Suppose that n is a positive integer, p, $q \in \{0\} \cup [1, \infty)$, not both of p, q are zero, $f \in L^{n+p}(\mu)$, $g \in L^{n+q}(\mu)$, and f and g are non-negative. Then*

$$\begin{aligned} &\int_X \int_X f(x)^p (f(x) + g(y))^n g(y)^q d\mu^2 \\ &\geq \left(\int_X f d\mu\right)^p \left(\int_X f d\mu + \int_X g d\mu\right)^n \left(\int_X g d\mu\right)^q. \end{aligned}$$

If neither f nor g is the zero function then if $n + p > 1$, equality implies that f is constant, and if $n + q > 1$, equality implies that g is constant.

Proof.

$$\begin{aligned} \int_X \int_X f(x)^p (f(x) &+ g(y))^n g(y)^q d\mu^2 \\ &= \int_X \int_X \sum_{k=0}^n \binom{n}{k} f(x)^{k+p} g(y)^{n-k+q} d\mu^2 \\ &= \sum_{k=0}^n \binom{n}{k} \int \int_{X^2} f(x)^{k+p} g(y)^{n-k+q} d\mu(x) d\mu(y) \\ &= \sum_{k=0}^n \binom{n}{k} \int_X f(x)^{k+p} d\mu(x) \int g(y)^{n-k+q} d\mu(y) \\ &\geq \sum_{k=0}^n \binom{n}{k} \left(\int_X f d\mu\right)^{k+p} \left(\int_X g d\mu\right)^{n-k+q} \\ &= \left(\int_X f d\mu\right)^p \left[\sum_{k=0}^n \binom{n}{k} \left(\int_X f d\mu\right)^k \left(\int_X g d\mu\right)^{n-k}\right] \left(\int_X g d\mu\right)^q \\ &= \left(\int_X f d\mu\right)^p \left(\int_X f d\mu + \int_X g d\mu\right)^n \left(\int_X g d\mu\right)^q, \end{aligned}$$

using the Lemma, from which the conditions for equality also descend. □

Remarks

1. Theorem 1 can be generalized to apply to two possibly different probability spaces, (X_i, μ_i), $i = 1, 2$; f will be a non-negative function on X_1, g will be a non-negative function on X_2, and the double integral will be an integral over the product space $(X_1 \times X_2, \mu_1 \times \mu_2)$. The proof is essentially unchanged.

2. The theorem can be further generalized to apply to a list of $k \geq 2$ probability spaces, (X_i, μ_i), $i = 1, \ldots, k$, with non-negative functions

$f_i \in L^{n+p_i}(\mu_i)$, and $p_i \in \{0\} \cup [1, \infty)$, $i = 1, \ldots, k$. In the proof, the multinomial expansion of $(f_1(x_1) + \cdots + f_k(x_k))^n$ plays the role of the binomial expansion in the proof of Theorem 1.

3. We are virtually certain that Theorem 1 and its generalizations hold true when n is allowed to be a non-integer in $(1, \infty)$, but we have no proof.