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Introduction of an Elementary Method to Express $\zeta(2k+1)$ in Terms of $\zeta(2k)$ with $k \geq 1$

Kazuyuki Fujii[1], **Tatsuo Suzuki**[2]

[1]Department of Mathematical Sciences
Yokohama City University
Yokohama, 236–0027
Japan

[2]Department of Mathematical Sciences
College of Systems Engineering and Science
Shibaura Institute of Technology
Saitama, 337–8570
Japan

email:fujii@yokohama-cu.ac.jp, suzukita@shibaura-it.ac.jp

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Abstract

In this note we give the most elementary method (as far as we know) to express $\zeta(2n+1)$ in terms of $\{\zeta(2k)|k \geq 1\}$. The method is based on only some elementary works by Leonhard Euler.

The zeta function $\zeta(x)$ is defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \quad \text{for} \quad x > 1$$

and we are interested in the (zeta–) values $\{\zeta(k) \mid k \geq 2\}$

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}. \tag{1}$$

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The value of $\zeta(2k)$ is well–known by Euler

$$\zeta(2)=\frac{\pi^2}{6},\quad \zeta(4)=\frac{\pi^4}{90},\quad \text{etc.} \tag{2}$$

while that of $\zeta(2k+1)$ is less–known (except for $\zeta(3)$ is irrational by Apéry). See [1] and its references. The textbook [2] is also recommended.
Therefore it is desirable to express $\zeta(2k+1)$ in terms of $\{\zeta(2k)|k\geq 1\}$ like

$$\zeta(2k+1)=c_0+\sum_{n=1}^{\infty}c_n\zeta(2n) \tag{3}$$

where $\{c_n|n\geq 0\}$ are constants.

In this note we revisit the problem and give a method by use of only a few elementary works by Euler.

Let us start by listing well–known works by Euler :

$$(a)\quad \sin x=\frac{e^{ix}-e^{-ix}}{2i}$$
$$(b)\quad \sin x=x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$$
$$(c)\quad \int_0^{\frac{\pi}{2}}x\log(\sin x)dx=-\frac{\pi^2}{8}\log 2+\frac{7}{16}\zeta(3)$$

Though equation (c) may be not popular among the Euler's works (which are huge ! [3]) it is important and interesting enough as shown in the following.

We begin by calculating the integral

$$\int_0^{\frac{\pi}{2}}x\log(\sin x)dx \tag{4}$$

in two ways by use of (a) and (b).

Here we list some well–known integrals related to (4)

$$\int_0^{\frac{\pi}{2}}\log(\sin x)dx=-\frac{\pi}{2}\log 2$$
$$\int_0^{\frac{\pi}{2}}\sin x\log(\sin x)dx=\log 2-1$$
$$\int_0^{\frac{\pi}{2}}x\log x dx=\frac{\pi^2}{8}\log\left(\frac{\pi}{2}\right)-\frac{\pi^2}{16}$$

for the convenience of readers.

(I) Calculation by use of (a)

By (a)

$$\log(\sin x) = \log(\frac{e^{ix} - e^{-ix}}{2i}) = \log(e^{ix} - e^{-ix}) - \log(2i) = \log\{e^{ix}(1 - e^{-2ix})\} - \log(2i)$$

$$= ix - \log(2i) + \log(1 - e^{-2ix}) = ix - \log(2i) - \sum_{n=1}^{\infty} \frac{e^{-2inx}}{n}, \tag{5}$$

where we have used the Taylor expansion

$$\log(1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{for} \quad |z| < 1.$$

Then

$$\int_0^{\frac{\pi}{2}} x \log(\sin x) dx = i \int_0^{\frac{\pi}{2}} x^2 dx - \log(2i) \int_0^{\frac{\pi}{2}} x dx - \sum_{n=0}^{\infty} \frac{1}{n} \int_0^{\frac{\pi}{2}} x e^{-2inx} dx$$

$$= i\frac{\pi^3}{24} - \frac{\pi^2}{8} \log(2i) - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\frac{\pi}{2}} x e^{-2inx} dx$$

and

$$\int_0^{\frac{\pi}{2}} x e^{-2inx} dx = \left[\frac{e^{-2inx}}{-2in} x\right]_0^{\frac{\pi}{2}} + \frac{1}{2in} \int_0^{\frac{\pi}{2}} e^{-2inx} dx$$

$$= \frac{i\pi}{4n} e^{-i\pi n} - \frac{1}{4n^2}(1 - e^{-i\pi n}) = \frac{i\pi}{4n}(-1)^n - \frac{1}{4n^2}(1 - (-1)^n).$$

Therefore

$$\int_0^{\frac{\pi}{2}} x \log(\sin x) dx = i\frac{\pi^3}{24} - \frac{\pi^2}{8} \log(2i) - \sum_{n=1}^{\infty} \frac{1}{n} \left\{\frac{i\pi}{4n}(-1)^n - \frac{1}{4n^2}(1 - (-1)^n)\right\}$$

$$= i\frac{\pi^3}{24} - \frac{\pi^2}{8} \log(2i) - \frac{i\pi}{4} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3}$$

$$= i\frac{\pi^3}{24} - \frac{\pi^2}{8} \log(2i) + \frac{i\pi}{4} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}.$$

Since it is easy to see

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{2}\zeta(2)$$

and

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}=\sum_{n=1}^{\infty}\frac{1}{n^3}-\sum_{n=1}^{\infty}\frac{1}{(2n)^3}=\frac{7}{8}\zeta(3)$$

we obtain

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x\log(\sin x)dx &= i\frac{\pi^3}{24}-\frac{\pi^2}{8}\log(2i)+\frac{i\pi}{4}\times\frac{1}{2}\zeta(2)+\frac{1}{2}\times\frac{7}{8}\zeta(3)\\ &= i\frac{\pi^3}{24}-\frac{\pi^2}{8}(\log 2+\log(i))+\frac{i\pi}{8}\zeta(2)+\frac{7}{16}\zeta(3)\\ &= -\frac{\pi^2}{8}\log 2+\frac{7}{16}\zeta(3)+i\left(\frac{\pi^3}{24}-\frac{\pi^3}{16}+\frac{\pi}{8}\zeta(2)\right)\\ &= -\frac{\pi^2}{8}\log 2+\frac{7}{16}\zeta(3)+i\frac{\pi}{8}\left(\zeta(2)-\frac{\pi^2}{6}\right), \end{aligned} \tag{6}$$

where we have used the principal value

$$\log(i)=\mathrm{Log}(e^{\frac{i\pi}{2}})=i\frac{\pi}{2}.$$

As a result we have $\zeta(2)=\frac{\pi^2}{6}$ **automatically** and equation (c).

(II) Calculation by use of (b)

By (b)

$$\log(\sin x)=\log x+\sum_{n=1}^{\infty}\log\left(1-\frac{x^2}{n^2\pi^2}\right), \tag{7}$$

so

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x\log(\sin x)dx &= \int_0^{\frac{\pi}{2}} x\log x dx+\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{2}} x\log\left(1-\frac{x^2}{n^2\pi^2}\right)dx\\ &= \frac{\pi^2}{8}\log\left(\frac{\pi}{2}\right)-\frac{\pi^2}{16}+\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{2}} x\log\left(1-\frac{x^2}{n^2\pi^2}\right)dx.\end{aligned}$$

Let us calculate the last term. By the change of variables ($x \longrightarrow n\pi\sqrt{x}$)

$$\int_0^{\frac{\pi}{2}} x\log\left(1-\frac{x^2}{n^2\pi^2}\right)dx=\frac{n^2\pi^2}{2}\int_0^{\frac{1}{4n^2}}\log(1-x)dx$$

and using the Taylor expansion

$$\log(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}$$

we obtain

$$\int_0^{\frac{\pi}{2}} x \log\left(1 - \frac{x^2}{n^2\pi^2}\right) dx = \frac{n^2\pi^2}{2}\left\{-\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\frac{1}{4n^2}} x^k dx\right\}$$

$$= -\frac{n^2\pi^2}{2}\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\frac{1}{4^{k+1}n^{2k+2}} = -\frac{\pi^2}{8}\sum_{k=1}^{\infty}\frac{1}{k(k+1)2^{2k}}\frac{1}{n^{2k}}.$$

As a result we have

$$\int_0^{\frac{\pi}{2}} x\log(\sin x)dx = \frac{\pi^2}{8}\log\left(\frac{\pi}{2}\right) - \frac{\pi^2}{16} - \frac{\pi^2}{8}\sum_{k=1}^{\infty}\frac{1}{k(k+1)2^{2k}}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\right)$$

$$= \frac{\pi^2}{8}\log\left(\frac{\pi}{2}\right) - \frac{\pi^2}{16} - \frac{\pi^2}{8}\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(k+1)2^{2k}}. \tag{8}$$

By comparing (I) with (II) we obtain the expression of $\zeta(3)$

$$\zeta(3) = \frac{2\pi^2}{7}\left\{\log\pi - \frac{1}{2} - \sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(n+1)2^{2n}}\right\}. \tag{9}$$

However, our expression (9) is of course not new, see for example [4] or [5].

By the way, the expression by Euler is different from ours :

$$\zeta(3) = \frac{\pi^2}{7}\left\{1 - 4\sum_{n=1}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}}\right\}, \tag{10}$$

see [1]. Therefore these two expressions give the (interesting) equation

$$\log\pi = 1 + \sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(2n+1)2^{2n}} \iff \log\left(\frac{\pi}{e}\right) = \sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(2n+1)2^{2n}}. \tag{11}$$

In the following we generalize the method above to obtain equation (3). For that purpose we consider the integral

$$\int_0^{\frac{\pi}{2}} x^{2l-1}\log(\sin x)dx \quad \text{for} \quad l \geq 1. \tag{12}$$

It may be reasonable to call this the **Euler integral**.

We calculate (12) in two ways by use of (a) and (b).

(I′) Calculation by use of (a)

In a similar way in (I) it is easy to see

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} x^{2l-1}\log(\sin x)dx &= i\int_0^{\frac{\pi}{2}} x^{2l}dx - \log(2i)\int_0^{\frac{\pi}{2}} x^{2l-1}dx - \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}} x^{2l-1}e^{-2inx}dx \\
&= i\frac{(\frac{\pi}{2})^{2l+1}}{2l+1} - \left(\log 2 + i\frac{\pi}{2}\right)\frac{(\frac{\pi}{2})^{2l}}{2l} - \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}} x^{2l-1}e^{-2inx}dx \\
&= i\frac{(\frac{\pi}{2})^{2l+1}}{2l+1} - i\frac{(\frac{\pi}{2})^{2l+1}}{2l} - \frac{(\frac{\pi}{2})^{2l}}{2}\log 2 - \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}} x^{2l-1}e^{-2inx}dx \\
&= -i\frac{(\frac{\pi}{2})^{2l+1}}{2l(2l+1)} - \frac{(\frac{\pi}{2})^{2l}}{2l}\log 2 - \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}} x^{2l-1}e^{-2inx}dx.
\end{aligned}
$$

In order to calculate the last term (which is not so easy) we make use of the trick. From

$$
\int_0^{\frac{\pi}{2}} e^{-2inx}dx = \frac{e^{-i\pi n}-1}{-2in} = -\frac{1}{2i}\{(e^{-i\pi n}-1)n^{-1}\}
$$

we differentiate the equation above $2l-1$ times with respect to n

$$
(-2i)^{2l-1}\int_0^{\frac{\pi}{2}} x^{2l-1}e^{-2inx}dx = -\frac{1}{2i}\left(\frac{d}{dn}\right)^{2l-1}\{(e^{-i\pi n}-1)n^{-1}\}
$$

$$
= -\frac{1}{2i}\left\{\sum_{j=0}^{2l-2}\frac{(2l-1)!}{j!(2l-1-j)!}(-i\pi)^{2l-1-j}e^{-i\pi n}\cdot(-1)^j j! n^{-j-1} - (e^{-i\pi n}-1)(2l-1)!n^{-2l}\right\}
$$

$$
= \frac{(2l-1)!}{-2i}\left\{\sum_{j=0}^{2l-2}\frac{(i\pi)^{2l-1-j}}{(2l-1-j)!}\frac{(-1)^{n-1}}{n^{j+1}} + \frac{1-(-1)^n}{n^{2l}}\right\}
$$

where we have used the Leibniz's rule of differentiation, so

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} x^{2l-1}e^{-2inx}dx &= \frac{(2l-1)!}{(-2i)^{2l}}\left\{\sum_{j=0}^{2l-2}\frac{(i\pi)^{2l-1-j}}{(2l-1-j)!}\frac{(-1)^{n-1}}{n^{j+1}} + \frac{1-(-1)^n}{n^{2l}}\right\} \\
&= \frac{(-1)^l(2l-1)!}{2^{2l}}\left\{\sum_{j=0}^{2l-2}\frac{(i\pi)^{2l-1-j}}{(2l-1-j)!}\frac{(-1)^{n-1}}{n^{j+1}} + \frac{1-(-1)^n}{n^{2l}}\right\}.
\end{aligned}
$$

By noting $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^k} = \left(1-\frac{1}{2^{k-1}}\right)\zeta(k), \quad \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^k} = 2\left(1-\frac{1}{2^k}\right)\zeta(k)$ we obtain

$$\int_0^{\frac{\pi}{2}} x^{2l-1}\log(\sin x)dx$$

$$= -i\frac{(\frac{\pi}{2})^{2l+1}}{2l(2l+1)} - \frac{(\frac{\pi}{2})^{2l}}{2l}\log 2 + \frac{(-1)^{l-1}(2l-1)!}{2^{2l}}\left\{\sum_{j=0}^{2l-2}\frac{(i\pi)^{2l-1-j}}{(2l-1-j)!}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{j+2}} + \sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^{2l+1}}\right\}$$

$$= -i\frac{(\frac{\pi}{2})^{2l+1}}{2l(2l+1)} - \frac{(\frac{\pi}{2})^{2l}}{2l}\log 2$$

$$+\frac{(-1)^{l-1}(2l-1)!}{2^{2l}}\left\{\sum_{j=0}^{2l-2}\frac{(i\pi)^{2l-1-j}}{(2l-1-j)!}\left(1-\frac{1}{2^{j+1}}\right)\zeta(j+2) + 2\left(1-\frac{1}{2^{2l+1}}\right)\zeta(2l+1)\right\}$$

(dividing the sum into $j = 2k$ $(k = 0, \cdots, l-1)$ and $j = 2k-1$ $(k = 1, \cdots, l-1)$)

$$= -i\frac{(\frac{\pi}{2})^{2l+1}}{2l(2l+1)} - \frac{(\frac{\pi}{2})^{2l}}{2l}\log 2$$

$$+\frac{(-1)^{l-1}(2l-1)!}{2^{2l}}\sum_{k=0}^{l-1}\frac{(i\pi)^{2l-1-2k}}{(2l-1-2k)!}\left(1-\frac{1}{2^{2k+1}}\right)\zeta(2k+2)$$

$$+\frac{(-1)^{l-1}(2l-1)!}{2^{2l}}\left\{\sum_{k=1}^{l-1}\frac{(i\pi)^{2(l-k)}}{(2l-2k)!}\left(1-\frac{1}{2^{2k}}\right)\zeta(2k+1) + \frac{2^{2l+1}-1}{2^{2l}}\zeta(2l+1)\right\}$$

$$= -\frac{(\frac{\pi}{2})^{2l}}{2l}\log 2 + \frac{(-1)^{l-1}(2l-1)!}{2^{2l}}\sum_{k=1}^{l-1}\frac{(-1)^{l-k}\pi^{2(l-k)}}{(2(l-k))!}\left(1-\frac{1}{2^{2k}}\right)\zeta(2k+1)$$

$$+(-1)^{l-1}(2l-1)!\frac{2^{2l+1}-1}{2^{4l}}\zeta(2l+1)$$

$$+i\left\{-\frac{(\frac{\pi}{2})^{2l+1}}{2l(2l+1)} + \frac{(-1)^{l-1}(2l-1)!}{2^{2l}}\sum_{k=0}^{l-1}\frac{(-1)^{l-k-1}\pi^{2(l-k)-1}}{(2(l-k)-1)!}\left(1-\frac{1}{2^{2k+1}}\right)\zeta(2k+2)\right\}$$

$$= -\frac{(\frac{\pi}{2})^{2l}}{2l}\log 2 + \frac{(2l-1)!}{2^{2l}}\sum_{k=1}^{l-1}\frac{(-1)^{k-1}\pi^{2(l-k)}}{(2(l-k))!}\left(1-\frac{1}{2^{2k}}\right)\zeta(2k+1)$$

$$+(-1)^{l-1}(2l-1)!\frac{2^{2l+1}-1}{2^{4l}}\zeta(2l+1)$$

$$+i\left\{-\frac{(\frac{\pi}{2})^{2l+1}}{2l(2l+1)} + \frac{(2l-1)!}{2^{2l}}\sum_{k=0}^{l-1}\frac{(-1)^{k}\pi^{2(l-k)-1}}{(2(l-k)-1)!}\left(1-\frac{1}{2^{2k+1}}\right)\zeta(2k+2)\right\}.$$

From this equation the imaginary part must be zero, so we have

$$\frac{(2l-1)!}{2^{2l}}\sum_{k=0}^{l-1}\frac{(-1)^k\pi^{2(l-k)-1}}{(2(l-k)-1)!}\left(1-\frac{1}{2^{2k+1}}\right)\zeta(2k+2)-\frac{(\frac{\pi}{2})^{2l+1}}{2l(2l+1)}=0 \quad (l=1,2,\cdots).$$

Here, if we assume

$$\zeta(0)=-\frac{1}{2} \tag{13}$$

then the equation above is rewritten in a compact form

$$\sum_{k=-1}^{l-1}\frac{(-1)^k\pi^{2(l-k)-1}}{(2(l-k)-1)!}\left(1-\frac{1}{2^{2k+1}}\right)\zeta(2k+2)=0 \quad (l=1,2,\cdots)$$

or $(k \longrightarrow k-1)$

$$\sum_{k=0}^{l}\frac{(-1)^{k-1}\pi^{2(l-k)+1}}{(2(l-k)+1)!}\left(1-\frac{1}{2^{2k-1}}\right)\zeta(2k)=0 \quad (l=1,2,\cdots).$$

From this we have the recurrent relation

Result 1 For $l=1,2,\cdots$

$$\zeta(2l)=\frac{2^{2l-1}}{2^{2l-1}-1}\sum_{k=0}^{l-1}\frac{(-1)^{l+k-1}\pi^{2(l-k)}}{(2(l-k)+1)!}\left(1-\frac{1}{2^{2k-1}}\right)\zeta(2k). \tag{14}$$

Let us list some examples :

$$\zeta(2)=\frac{\pi^2}{6},\quad \zeta(4)=\frac{\pi^4}{90},\quad \text{etc.}$$

Next, from the real part of the equation we have

Result 2 For $l=1,2,\cdots$

$$\int_0^{\frac{\pi}{2}}x^{2l-1}\log(\sin x)dx=-\frac{(\frac{\pi}{2})^{2l}}{2l}\log 2+\frac{(2l-1)!}{2^{2l}}\sum_{k=1}^{l-1}\frac{(-1)^{k-1}\pi^{2(l-k)}}{(2(l-k))!}\left(1-\frac{1}{2^{2k}}\right)\zeta(2k+1)$$
$$+(-1)^{l-1}(2l-1)!\frac{2^{2l+1}-1}{2^{4l}}\zeta(2l+1). \tag{15}$$

Let us list some examples :

$$\int_0^{\frac{\pi}{2}}x\log(\sin x)dx=-\frac{\pi^2}{8}\log 2+\frac{7}{16}\zeta(3),$$
$$\int_0^{\frac{\pi}{2}}x^3\log(\sin x)dx=-\frac{\pi^4}{64}\log 2+\frac{9\pi^2}{64}\zeta(3)-\frac{93}{128}\zeta(5),\quad \text{etc.}$$

(II′) Calculation by use of (b)

In a similar way in (II) it is easy to see

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} x^{2l-1}\log(\sin x)dx &= \int_0^{\frac{\pi}{2}} x^{2l-1}\log x dx + \sum_{n=1}^{\infty}\int_0^{\frac{\pi}{2}} x^{2l-1}\log\left(1-\frac{x^2}{n^2\pi^2}\right)dx \\
&= \frac{(\frac{\pi}{2})^{2l}}{2l}\log\left(\frac{\pi}{2}\right) - \frac{(\frac{\pi}{2})^{2l}}{(2l)^2} + \sum_{n=1}^{\infty}\frac{(n^2\pi^2)^l}{2}\int_0^{\frac{1}{4n^2}} t^{l-1}\log(1-t)dt \\
&= \frac{(\frac{\pi}{2})^{2l}}{2l}\log\left(\frac{\pi}{2}\right) - \frac{(\frac{\pi}{2})^{2l}}{(2l)^2} - \sum_{n=1}^{\infty}\frac{(n^2\pi^2)^l}{2}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\frac{1}{4n^2}} t^{k+l-1}dt \\
&= \frac{(\frac{\pi}{2})^{2l}}{2l}\log\left(\frac{\pi}{2}\right) - \frac{(\frac{\pi}{2})^{2l}}{(2l)^2} - \sum_{n=1}^{\infty}\frac{(n^2\pi^2)^l}{2}\sum_{k=1}^{\infty}\frac{1}{k(k+l)(4n^2)^{k+l}} \\
&= \frac{(\frac{\pi}{2})^{2l}}{2l}\log\left(\frac{\pi}{2}\right) - \frac{(\frac{\pi}{2})^{2l}}{(2l)^2} - \sum_{n=1}^{\infty}\frac{n^{2l}\pi^{2l}}{2}\sum_{k=1}^{\infty}\frac{1}{k(k+l)2^{2(k+l)}n^{2k}n^{2l}} \\
&= \frac{(\frac{\pi}{2})^{2l}}{2l}\log\left(\frac{\pi}{2}\right) - \frac{(\frac{\pi}{2})^{2l}}{(2l)^2} - \left(\frac{\pi}{2}\right)^{2l}\cdot\frac{1}{2}\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(k+l)2^{2k}} \\
&= \left(\frac{\pi}{2}\right)^{2l}\left\{\frac{1}{2l}(\log\pi-\log 2) - \frac{1}{(2l)^2} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(k+l)2^{2k}}\right\}.
\end{aligned}
$$

Therefore we have

Result 3 For $l = 1, 2, \cdots$

$$
\int_0^{\frac{\pi}{2}} x^{2l-1}\log(\sin x)dx = \left(\frac{\pi}{2}\right)^{2l}\left\{\frac{1}{2l}(\log\pi-\log 2) - \frac{1}{(2l)^2} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(k+l)2^{2k}}\right\}. \tag{16}
$$

By comparing (15) with (16) we have the main result

Result 4 (Main) For $l = 1, 2, \cdots$

$$
\begin{aligned}
\zeta(2l+1) = \frac{(-1)^l 2^{2l}}{2^{2l+1}-1}&\left\{\sum_{k=1}^{l-1}\frac{(-1)^{k-1}\pi^{2(l-k)}}{(2(l-k))!}\left(1-\frac{1}{2^{2k}}\right)\zeta(2k+1)\right. \\
&\left. -\frac{\pi^{2l}}{(2l)!}\left(\log\pi - \frac{1}{2l} - l\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(k+l)2^{2k}}\right)\right\}. \qquad (17)
\end{aligned}
$$

For examples,

$$\zeta(3)=\frac{2\pi^2}{7}\left\{\log\pi-\frac{1}{2}-\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(k+1)2^{2k}}\right\},$$

$$\begin{aligned}\zeta(5)&=\frac{6\pi^2}{31}\left\{\zeta(3)-\frac{\pi^2}{9}\left(\log\pi-\frac{1}{4}-2\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(k+2)2^{2k}}\right)\right\}\\&=\frac{4\pi^4}{651}\left\{\frac{11}{2}\log\pi-\frac{29}{8}-\sum_{k=1}^{\infty}\frac{(2k+11)\zeta(2k)}{k(k+1)(k+2)2^{2k}}\right\}.\end{aligned}$$

A comment is in order. There are many expressions like (17). For example, in [6] (Theorem A) it is given

$$\zeta(2l+1)=(-1)^l\frac{(2\pi)^{2l}}{l(2^{2l+1}-1)}\left[\sum_{k=1}^{l-1}(-1)^{k-1}\frac{k\zeta(2k+1)}{\pi^{2k}(2l-2k)!}+\sum_{k=1}^{\infty}\frac{\zeta(2k)(2k)!}{2^{2k}(2k+2l)!}\right].$$

However, our expression is different from this.

In this note we gave a simple method to obtain some deep relations among zeta–values by calculating the Euler integral (in our terminology). The method is systematic and most elementary as far as we know.

References

[1] WolframMathWorld : Riemann Zeta Function, http://mathworld.wolfram.com/RiemannZetaFunction.html

[2] M. Koecher : Klassische elementare Analysis, Birkhäuser Basel–Boston, 1987.

[3] W. Dunham : Euler : The Master of Us All, Mathematical Association of America, 1999.

[4] M. Sugioka : http://www5b.biglobe.ne.jp/~sugi_m/page021.htm

[5] I. Sato : http://www.geocities.jp/ikuro_kotaro/index.htm

[6] D. Cvijović and J. Klinowski : New Rapidly Convergent Series Representations for $\zeta(2n+1)$, Proc. Amer. Math. Soc., **125** (1997), 1263–1271.