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A Note on Stable Matchings

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Abstract

The Gale-Shapley Theorem, sometimes called the Stable Marriage Theorem, asserts that there is a stable matching in every finite locally ordered bipartite graph. We show that this statement characterizes bipartite graphs, among finite simple graphs.

All graphs referred to here will be finite and simple. The vertex set of a graph G is denoted $V(G)$, and the edge set is $E(G)$. An edge with vertices u and v at its ends can be denoted uv, or vu. For each $u \in V(G)$ let E_u denote the set of edges of G which are incident to u. Since G is simple, E_u is in obvious one-to-one correspondence with $N(u) = \{v \in V(G) | uv \in E(G)\}$.

A *local ordering* on a graph G is an indexed collection $L = [<_u; u \in V(G)]$ in which, for each $u \in V(G)$, $<_u$ is a linear ordering on E_u. A matching (set of independent edges) $M \subseteq E(G)$ is *stable* with respect to a local ordering L on G if and only if for each edge $uv \in E(G) \backslash M$, either there is an edge $u\,x \epsilon M$ such that $uv <_u ux$, or there is an edge $vy \in M$ such that $uv <_v vy$.

Some explanation may be welcome! Think of M as a pairing of vertices for some purpose, such as marriage, or working in pairs on some project. Think of $uv <_u ux$ as saying: u prefers x over v as a partner. Then the matching M is *not* stable if and only if there exists an edge (possible partnership) uv of G, which is not in M, such that neither u nor v is paired by M with a

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partner candidate that they prefer over v or u, respectively. The description of M as "unstable" because of the existence of such an edge $uv \in E(G)\backslash M$ is obviously inspired by elementary postulates regarding human nature, but–who knows?–the concept of a stable matching may one day find application in chemistry, or particle physics.

In the original formulation of Gale and Shapley [1] the pairings given by the matching were thought to be pairings of objects from two different categories, as, for instance, men and women in marriages, or teachers and classes (groups of students) that they might be assigned to teach. So it was natural to confine the inquiry to bipartite graphs G. Their celebrated result may be stated thus.

Theorem 1. *If G is bipartite graph, then for any local ordering L on G there is a stable matching in G with respect to L.*

Our note is the following.

Theorem 2. *If G is a non-bipartite graph then there is a local ordering on G with respect to which there is no stable matching in G.*

Proof. Since G is non-bipartite, G has an odd cycle C_q as an induced subgraph. Let the vertices of C_q be $u_1, \ldots, u_q$, in one order around the cycle. Reading subscripts mod q, let L be any local ordering on G satisfying: u_i prefers u_{i-1} and u_{i+1} to any of its neighbors off C_q, and u_i prefers u_{i+1} to u_{i-1}.

If M is a matching in G, then, because C_q is an odd cycle, there exists $i \in \{1, \ldots, q\}$ such that neither $u_{i-1}u_i$ nor u_iu_{i+1} is in M. Then $u_{i-1}u_i \in E(G)\backslash M$ is a "bad" edge for M: neither is there an edge $u_{i-1}x \in M$ such that u_{i-1} prefers x to u_i, nor is there an edge $u_iy \in M$ such that u_i prefers y to u_{i-1}. □

References

[1] D. Gale and L. S. Shapley, College admissions and the stability of marriage, *The American Mathematical Monthly* 120 (no. 5, May, 2013), 386-391; this paper originally appeared in the AMM in 1962.