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The completed zeta function and the Riemann Hypothesis

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Abstract

The famous Riemann Hypothesis asserts that all the non-trivial zeros of the zeta function have real part $\frac{1}{2}$. Based on some recent computer calculations showing that the first discovered 10 trillion non-trivial zeros, ordered by increasing positive imaginary part, have real part $\frac{1}{2}$, it may be worthwhile to look at some important consequences if the Riemann Hypothesis is true. We then phrase the Riemann hypothesis in terms of the completed zeta function rather than the zeta function. We finally find results in that direction from one of which, to my pleasant surprise, Wallis Formula follows as an easy corollary.

1 Introduction

Let $z = \sigma + it$. For $\sigma > 1$, the **Riemann zeta function** ζ is defined by $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$. Now $|\frac{1}{n^z}| = \frac{1}{|e^{z \log n}|} = \frac{1}{|e^{\sigma \log n}|} = \frac{1}{n^\sigma}$. By Weierstrass test, the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly in the half-plane $\sigma > 1$ and hence on every compact subset of this half-plane. Thus ζ is analytic in the half-plane

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$\sigma > 1$ being the sum function of a uniformly convergent series of analytic functions. With some work, this function can be continued analytically to all complex $z \neq 1$. As a result, the zeta function is analytic everywhere except for a simple pole at $z = 1$ with residue 1. Even though Bernhard Riemann's 1859 condensed 8-page paper [22] was his only work spanning Number Theory since his preoccupation was developing the theory of complex functions (he emphasized the geometric aspects of the theory in contrast to the purely analytic approach taken by another co-founder, Cauchy (1789-1857)), it had a deep impact on Mathematics and in particular on Analytic Number Theory of which we mention $\prod_{p \text{ prime}} \frac{1}{1-\frac{1}{p^z}} = \sum_{n=1}^{\infty} \frac{1}{n^z}$ previously discovered by Euler but for real z. In other words, the Riemann zeta function is not only important as a function of a complex variable but also contains information about prime numbers and their distribution.
Riemann's defined zeta function can now be related to Bernoulli numbers $B_k, k = 0, 1, 2, 3...$ defined by $\frac{z}{e^z-1} = \sum_0^{\infty} B_k \frac{z^k}{k!}$ as in Euler famous identity [11]:

$$\zeta(2k) = \frac{(-1)^{k-1} 2^{2k-1} B_{2k} \pi^{2k}}{(2k)!}$$

and [9] if k is a positive integer, then $\zeta(-k) = -\frac{B_{k+1}}{k+1}$. In particular, the latter case implies that $\zeta(-2k) = 0$ (see [10], p. 21 for a proof). The negative even integers are called **trivial zeros** of the zeta function. The other zeros (there are plenty as we shall see later) are called the **non-trivial zeros** and we'll show that they are confined INSIDE what is known as the **Critical Strip** $\{z : 0 \leq Rez \leq 1\}$. To see this, first note that the Euler Product Formula implies that there are no zeros of $\zeta(z)$ with real part > 1 since convergent infinite products never vanish. Next, using Riemann Functional Equation $\pi^{-\frac{z}{2}}\Gamma(\frac{z}{2})\zeta(z) = \pi^{-\frac{1-z}{2}}\Gamma(\frac{1-z}{2})\zeta(1-z)$ and the fact that Γ has no zeros in $\mathbb{C}$, it follows that there are no zeros of $\zeta(z)$ with real part < 0 apart from $..., -6, -4, -2$. Finally, using $\zeta(1+it) \neq 0, \forall t \in \mathbb{R}$ (More details about this coming up) and the Functional Equation again, the result follows.

In the same paper Riemann conjectured that ALL non-trivial zeros of his zeta function have real part equal to $\frac{1}{2}$ (Recent computer calculations have shown that the first discovered 10 trillion non-trivial zeros, ordered by increasing positive imaginary part, lie on the critical line $\frac{1}{2} + it$, where t is a real number; the approximate values of t for the first 6 zeros are $14.13472, 21.02203, 25.01085, 30.42487, 32.93506$, and 37.58617). This has been known as the **Riemann Hypothesis**.

Now, we elaborate on $\zeta(1+it) \neq 0$ which is essential for an analytic proof of a deep theorem known as the **Prime Number Theorem**:
On the basis of counting primes, one may be led to suspect that the number of primes less than or equal to a positive number x, denoted by $\pi(x)$, increases somehow like $\frac{x}{\log x}$. As a matter of fact, in 1791 at the age of 14, Gauss conjectured that $\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1$. In 1850, trying to settle the Gauss conjecture, Tchebycheff showed that there exist positive constants c and C such that

$$c\frac{x}{\log x} < \pi(x) < C\frac{x}{\log x}$$

for $x \geq 2$ with $c = .92$ and $C = 1.11$
In 1859, while attempting to find a formula for $\pi(x)$, Riemann (a student of Gauss) discovered analytic properties of his zeta function.
Throughout the last decade of the nineteenth century, J. Hadamard became interested in Gauss conjecture and the result was his theory of entire functions. It was not until 1896 that the Gauss conjecture was settled by Hadamard and, simultaneously by, de la Vallée Poussin and from then on it has been known as the **Prime Number Theorem.** Both Hadamard and de la Vallée Poussin proved that $\zeta(1+it) \neq 0$ from which they deduced the Prime Number Theorem.

Remark. Hadamard and de la Vallée Poussin also showed the converse to be true and for a while it appeared that the Prime Number Theorem was impossible to prove without using $\zeta(1+it) \neq 0$. However, in 1949, Erdös and Selberg proved the Prime Number Theorem by "elementary" methods meaning without using functions of a complex variable. Below we state Hadamard and de la Vallée Poussin key result:

Theorem 1.1. $\zeta(1+it) \neq 0, \forall t \in \mathbb{R}$. *That is, no zeros of the zeta function could lie on the line* $1+it$.

2 Consequences

As we mentioned in the previous section it is deeply felt that all analytic proofs of the Prime Number Theorem rely on $\zeta(1+it) \neq 0$. Thus, should the Riemann Hypothesis be true, the Prime Number Theorem would be, of course, a consequence of the hypothesis as $\zeta(1+it) \neq 0$ would then follow trivially.

Definition 2.1. *Let p be a positive integer. A* **Dirichlet Character modulo** *p is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with the following conditions*
1) $\chi(m+p) = \chi(m) \forall m \in \mathbb{Z}$.
2) $\forall m \in \mathbb{Z}$, we have

$$\begin{cases} \chi(m) \neq 0 & if \ (m,p) = 1 \\ \chi(m) = 0 & if \ (m,p) > 1 \end{cases}$$

3) $\chi(m_1 m_2) = \chi(m_1)\chi(m_2) \forall m_1, m_2 \in \mathbb{Z}$.

Generalized Riemann Hypothesis. For every Dirichlet character χ and every complex number z with $\sum_1^\infty \frac{\chi(n)}{n^z} = 0$ if the real part of z is between 0 and 1, then it is actually $\frac{1}{2}$.

Remark 2.2. *The Generalized Riemann Hypothesis clearly implies the Riemann Hypothesis as we can specialize by taking $p = 1$ and $\chi(n) = 1, \forall n \in \mathbb{Z}$.*

In a letter to Euler in 1742, Christian Goldbach conjectured that:
Goldbach Conjecture. Every natural number $n > 5$ is a sum of three primes.
Euler believed it and phrased it as:
Every even natural number $n \geq 4$ is a sum of two primes.

Nowadays, Goldbach conjecture is split into:
Strong Goldbach conjecture. Every EVEN natural number $n > 2$ is the sum of two primes.
Weak Goldbach conjecture. Every ODD natural number $n > 5$ is the sum of three primes.
As the above terminology suggests, the strong Goldbach conjecture implies the weak Goldbach conjecture because, given an odd natural number $n > 5$, we subtract 3 from n resulting in an EVEN natural number $n - 3$ which is the sum of two primes.

Hardy and Littlewood took up the Goldbach conjecture and proved the following theorems in the years 1922 and 1924 respectively:

Theorem 2.3. *[14] If the Generalized Riemann Hypothesis is true, then almost all EVEN numbers are sums of two primes.*

More specifically, if $E(K)$ is the number of even integers $k < K$ that are not a sum of two primes, then for any $\epsilon > 0$ we have $E(K) = O(K^{\frac{1}{2}+\epsilon})$.

Theorem 2.4. *[15] If the Generalized Riemann Hypothesis is true, then every ODD natural number is a sum of three ODD primes for sufficiently large $n > n_0$.*

In 1926, Bruno Lucke [21] calculated from [15] that "sufficiently large" involves $n_0 = 10^{32}$ and in 1997 Dimitrii Zinoviev [28] obtained an excellent result refining that to 10^{20}. Later that year, with more analysis and extensive computation the following interesting result showing a close relation between the Generalized Riemann Hypothesis and the Goldbach Conjecture has finally materialized:

Theorem 2.5. *[8] If the Generalized Riemann Hypothesis is true, then every ODD number $n > 5$ is the sum of three primes.*

It would be a miss here not to mention an important 1937 related result by Vinogradov without using the Generalized Riemann Hypothesis:

Theorem 2.6. *(Vinogradov Theorem) [27] Every sufficiently large ODD number $n > n_o$ is the sum of three primes.*

However, an elaborate study of the proof of the above theorem reveals some bad news. Even though Vinogradov proved it without using the Generalized Riemann Hypothesis, Borodzkin [4] calculated that "sufficiently large" involves $n_o = 3^{3^{15}}$ which is approximately $10^{7000000}$ which even though this has been refined by Chen and Wang [7] to e^{99012}, and later by Liu and Wang [20] to e^{3100}, it is still too large for cases to be checked by computers at the time of this writing.
On the other hand, the following is a surprising and recent result (May 13, 2013) in a 133-page paper by Harald Helfgott [17] which we can now call the Weak Goldbach Theorem:

Theorem 2.7. *Every odd number > 5 is the sum of 3 primes.*

Indeed, this unconditional result was proven, by Helfgott, for odd numbers $> 10^{30}$ but the result has been checked by computer for odd numbers $< 10^{30}$. Despite the importance of the previous result, the Goldbach Strong Conjecture is still unsettled.

Remark 2.8. *In England during the early years of the twentieth century, many mathematicians thought that the Riemann Hypothesis was not such a difficult problem. As a matter of fact, Ernest W. Barnes assigned it to his student J. E. Littlewood as a research problem for his doctoral dissertation at Cambridge. Even when realizing its apparent difficulty, Littlewood did not regret tackling it and stated:*
"Try a hard problem. You may not solve it, but you will prove something else".
This philosophy worked well with him for he proved numerous results afterwards, most as a joint work with G.H. Hardy. Hardy [16], by the way, proved an important result in 1914, *that an infinite number of zeros of the zeta function lie on the critical line $x = \frac{1}{2}$. Unfortunately, however, that is short of proving that there did not exist other zeros of the zeta function that were not on the line.*

3 The completed zeta function

Definition 3.1. *The* **completed zeta function** *(or generically the xi-function), originally defined by Riemann [22], is*

$$\xi(z) = \frac{1}{2}z(z-1)\pi^{-\frac{z}{2}}\Gamma(\frac{z}{2})\zeta(z)$$

(Mathematically, the Gamma function was needed to complete the zeta function. Symbolically, ζ zeta; ξ completed).

Definition 3.2. *Let $f(z)$ be an entire function. The maximum modulus function, denoted by $M(r)$, is defined by $M(r)$ =max$\{|f(z)| : |z| = r\}$.*

Definition 3.3. *Let $f(z)$ be a non-constant entire function. The order ρ of $f(z)$ is defined by*

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

The order of any constant function is 0, *by convention.*

Definition 3.4. *An entire function $f(z)$ of positive order ρ is said to be of type τ if*

$$\tau = \limsup_{r\to\infty} \frac{\log M(r)}{r^\rho}.$$

Theorem 3.5. *[9] The following are some important properties of the completed zeta function:*

1. *$\xi(z) = \xi(1-z)$. This Functional Equation shows that the function $\xi(z)$ is symmetric about the critical line $Re(z) = \frac{1}{2}$.*
2. *The function $\xi(z)$ is entire.*
3. *The function $\xi(z)$ is of order one and infinite type.*
4. *The function $\xi(z)$ has infinitely many zeros.*

Remark 3.6. *It is clear now that the completed zeta function ξ is more convenient to use instead of the zeta function ζ since using the definition of ξ removes the simple pole of ζ at $z = 1$ and as a result the theory of entire functions can be applied, if needed, to ξ (Property 2 in the preceding theorem). In addition, since none of the factors of ξ except ζ has a zero in $\mathbb{C} - \{0, 1\}$, no information is lost about the non-trivial zeros.*

The Riemann Hypothesis can therefore be stated as:
The Riemann Hypothesis using the completed zeta function. ALL zeros of $\xi(z)$ are on the critical line $Re(z) = \frac{1}{2}$.

Remark 3.7. *It is impossible for ξ to become 0 at the trivial zeros $-2k, k = 1, 2, 3, ...$ because otherwise, using $\xi(1-z) = \xi(z)$, we get $\xi(2k+1) = 0$ and therefore, using the definition of the the completed zeta function, yields $\zeta(2k+1) = 0$, which is impossible since all the zeros of ζ are on the negative even integers or inside the critical strip.*

Remark 3.8. *As of the time of this writing, no double zero of the function ξ has been found on the critical line.*

4 Results

We begin with the following results

Theorem 4.1. *[9] Let T denote the set of zeros of the completed zeta function ξ in the critical strip whose real part is greater than $\frac{1}{2}$. Then $\frac{\xi'(z)}{\xi(z)}$ is analytic on T if and only if the Riemann Hypothesis is true.*

Theorem 4.2. *(see for example [23])* $Re(\frac{\xi'(z)}{\xi(z)}) > 0$ *on* $\{z : Re(z) > 1/2\}$ *if and only if the Riemann Hypothesis is true.*

Remark 4.3. *Recall that the Gamma function* Γ *played a major role in the functional equation* $\pi^{-\frac{z}{2}}\Gamma(\frac{z}{2})\zeta(z) = \pi^{-\frac{1-z}{2}}\Gamma(\frac{1-z}{2})\zeta(1-z)$ *to determine the location of the trivial zeros of the zeta function* ζ*. By analogy, the Digamma function* $\Psi(z) := (\log\Gamma(z))' = \frac{\Gamma'(z)}{\Gamma(z)}$ *appears ([18], Equation (3.8)) in* $\frac{\xi'(z)}{\xi(z)}$*, the logarithmic derivative of the completed zeta function:*

$$\frac{\xi'(z)}{\xi(z)} = \frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1} + \frac{1}{2}\frac{\Gamma'(\frac{z}{2})}{\Gamma(\frac{z}{2})} + \frac{1}{z} - \frac{1}{2}\log\pi.$$

The following is another characterization of the Riemann Hypothesis in terms of the completed zeta function

Theorem 4.4. *[19] The Riemann Hypothesis is true iff* $\lambda_k := \frac{1}{(k-1)!}\frac{d^k[z^{k-1}\log\xi(z)]}{dz^k}\Big|_{z=1} \geq 0$ *for every positive integer* k*.*

Remark 4.5. *It can be shown, using the definition of* ξ*, that* $\overline{\xi(z)} = \xi(\overline{z})$*. Suppose that* z *is a zero of* ξ*. Then* $\xi(z) = 0 = \overline{0} = \overline{\xi(z)} = \xi(\overline{z})$ *and so* $\overline{z}$ *is also a zero of* ξ*. Using the Functional Equation* $\xi(1-z) = \xi(z)$*, it follows that* $1-z$ *and* $1-\overline{z}$ *are also zeros of* ξ*.*
Now, the factor $\frac{1}{(k-1)!}$ *is intentionally retained due to the following connection [19]:*

$$\lambda_k = \sum_{z_n}[1-(1-\frac{1}{z_n})^k],$$

where the sum is taken over the non-trivial zeros $\{z_n\}$ *of the zeta function* ζ *(with* z_n *and* $1-z_n$ *being paired together). Using ([5], lemma 2.2)*

$$\gamma - c(x-1) < \frac{\zeta'(x)}{\zeta(x)} + \frac{1}{x-1} < \gamma$$

for $x > 1$ *and some constant* c *with* γ *being Euler's constant, it follows that* $\lim_{x\to1^+}\frac{\zeta'(x)}{\zeta(x)} + \frac{1}{x-1} = \gamma$*. In addition, ([6], Equation 6.3.3),* $\frac{1}{2}\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = -\frac{\gamma}{2} - \log 2$*. Consequently, using the previous representation of* $\frac{\zeta'(x)}{\zeta(x)}$*, we get*

$$\frac{\xi'(1)}{\xi(1)} = \frac{\gamma}{2} - \log 2 + 1 - \frac{\log\pi}{2}.$$

If we now take the special case $k = 1$ we can write the following interesting result for the sum of the reciprocals of the non-trivial zeros of the zeta function:

Theorem 4.6.
$$\sum_{z_n} \frac{1}{z_n} = \frac{\gamma}{2} - \log 2 + 1 - \frac{\log \pi}{2}$$
where the sum is taken over the non-trivial zeros $\{z_n\}$ of the zeta function ζ (with the understanding that z_n and $1 - z_n$ are being paired together).

Next we look at the following interesting reformulation of the Riemann Hypothesis due to Warren D. Smith [25] in which he used the conformal map $w = 1 - \frac{1}{z}$ to transform the region $Re(z) > \frac{1}{2}$ into the unit disk $|w| < 1$ via the function
$$F(w) = \log[\frac{w}{1-w}\zeta(\frac{1}{1-w})]$$
thereby obtaining the following characterization of the Riemann Hypothesis in terms of a variant of the zeta function:

Theorem 4.7. *The Riemann hypothesis is true iff $F(w) = \log[\frac{w}{1-w}\zeta(\frac{1}{1-w})]$ is analytic in $|w| < 1$.*

Remark 4.8. *Clearly, the above equivalence can be replaced with the statement that there exist complex constants $a_1, a_2, ...$ (which can actually be proven real; for instance $a_0 = 0, a_1 = \gamma$) such that the Maclaurin series $F(z) = \sum_0^\infty a_n w^n$ converges in $|w| < 1$.*

Using an appropriate Functional Equation of $\zeta(w)$ (see for instance [9]) we obtain the following

Theorem 4.9. *(Functional Equation of $F(w)$)*

$$F(w) = F(\frac{1}{w}) + \log(-\frac{w}{\pi}) + \frac{1}{1-w}\log(2\pi) + \log\sin\frac{\pi}{2(1-w)} + \log\Gamma(-\frac{w}{1-w}).$$

We add the following contribution which connects the functions F and ξ: Using the definitions of F and ξ we get

$$\log[\xi(\frac{1}{1-w})] = F(w) - \log(1-w) - \frac{1}{2(1-w)}\log\pi + \log\Gamma[\frac{1}{2(1-w)}]$$

which, upon exponentiation, yields

$$\xi(\frac{1}{1-w}) = e^{F(w)}\frac{1}{1-w}\pi^{-\frac{1}{2(1-w)}}\Gamma[\frac{1}{2(1-w)}]$$

and going back, $z = \frac{1}{1-w}$ gives

$$\xi(z) = z\pi^{-\frac{z}{2}}\Gamma(\frac{z}{2})e^{F(1-\frac{1}{z})}$$

and the question now is whether we can use analyticity of $\xi(z)$ everywhere in $\mathbb{C}$ to get analyticity of $F(z)$ in $|z| < 1$ which would prove that the Riemann Hypothesis is true.
We mention the following interesting result by Speiser [26]:

Theorem 4.10. *The Riemann Hypothesis is true if and only if ζ' has no zeros in the strip $\{z : 0 < Re(z) < \frac{1}{2}\}$. Thus the zeta function ζ has only simple zeros on the critical line if and only if its derivative ζ' has no zeros on the critical line.*

We now find a canonical representation of $\xi(z)$:
By Theorem 3.5, $\xi(z)$ is an entire function of order one and infinite type. Since the zeta function $\zeta(z)$ has a simple pole with residue 1 at $z = 1, \xi(1) = \frac{1}{2}$. Now, using the functional equation $\xi(z) = \xi(1-z), \xi(0) = \frac{1}{2}$ Then, by ([11] p. 47), we have

$$\xi(z) = e^A e^{Bz}\prod_{n=1}^{\infty}(1-\frac{z}{z_n})\exp\frac{z}{z_n},$$

where again $\{z_n\}_1^{\infty}$ are the non-zero zeros of $\xi(z)$ and hence, using the functional equation and the definition of the completed zeta function, are the non-trivial zeros of $\zeta(z)$ which are indeed in the critical strip and A and B are complex constants.
Now $\xi(0) = \frac{1}{2}$ implies that $e^A = \frac{1}{2}$ and so we can write

$$\xi(z) = \frac{1}{2}e^{Bz}\prod_{n=1}^{\infty}(1-\frac{z}{z_n})\exp\frac{z}{z_n}.$$

The effort to find B using $\xi(1) = \frac{1}{2}$ leads to

$$1 = e^b\prod_{n=1}^{\infty}(1-\frac{1}{z_n})e^{\frac{1}{z_n}}.$$

Consider the product

$$p = \prod_{n=1}^{\infty}(1 - \frac{1}{z_n})e^{\frac{1}{z_n}}.$$

Then

$$p^z = \prod_{n=1}^{\infty}(1 - \frac{1}{z_n})^z e^{\frac{z}{z_n}}.$$

Therefore,

$$\xi(z) = \frac{1}{2}e^{Bz}p^z \prod_{n=1}^{\infty}(1 - \frac{z}{z_n})(1 - \frac{1}{z_n})^{-z}.$$

Now $\xi(1) = \frac{1}{2}$ implies that $e^B p = 1$ and our identity reduces to the following representation of $\xi(z)$:

$$\xi(z) = \frac{1}{2}\prod_{n=1}^{\infty}(1 - \frac{z}{z_n})(1 - \frac{1}{z_n})^{-z}.$$

Remark 4.11. *It is worthwhile to draw the following analogy with $\frac{1}{\Gamma(z)}$ which is an entire function of order one and infinite type ([11], p.* 51*) and which is represented as ([11], p.* 49*):*

$$\frac{1}{\Gamma(z)} = z\prod_{n=1}^{\infty}(1 + \frac{z}{n})(1 + \frac{1}{n})^{-z}.$$

Using the product representations of $\xi(z)$ and $\frac{1}{\Gamma(z)}$ with the definition of $\xi(z)$ give the following explicit representation for $\zeta(z)(\{z_n\}$ denotes the sequence of non-trivial zeros of $\zeta(z))$:

Theorem 4.12.

$$\zeta(z) = \underbrace{\frac{1}{z-1}}_{\textit{for singularity}} \pi^{\frac{z}{2}} \prod_{n=1}^{\infty} \underbrace{(1 + \frac{z}{2n})}_{\textit{for trivial zeros}} \underbrace{(1 - \frac{z}{z_n})}_{\textit{for non-trivial zeros}} (1 + \frac{1}{n})^{-\frac{z}{2}}(1 - \frac{1}{z_n})^{-z}.$$

Corollary 4.13. Wallis Product

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}...$$

Proof. For $z \in \mathbb{C} - \{0,1\}$, we can rewrite the representation in the theorem as:

$$(z-1)\zeta(z) = \frac{\pi^{\frac{z}{2}}}{\frac{z}{2}} \underbrace{\prod_{n=1}^{\infty}(1-\frac{z}{z_n})(1-\frac{1}{z_n})^{-z}}_{2\xi(z)\text{ entire hence continuous}} \underbrace{\frac{z}{2}\prod_{n=1}^{\infty}(1+\frac{z}{2n})(1+\frac{1}{n})^{-\frac{z}{2}}}_{\frac{1}{\Gamma(\frac{z}{2})}\text{ entire hence continuous}}.$$

The result now follows using $\lim_{z\to 1}(z-1)\zeta(z) = 1$.
Interestingly enough, we easily obtain a similar corollary

Corollary 4.14.

$$\frac{\pi}{6} = \prod_{n=1}^{\infty} \frac{z_n}{z_n - 1}\frac{z_n - 2}{z_n - 1}$$

(Note that, with $\{z_n\}$ denoting the sequence of non-trivial zeros of $\zeta(z)$, $z_n \neq 1$ because $\zeta(1+it) \neq 0$ for all real t).

Proof.

$$\zeta(2) = \pi\prod_{n=1}^{\infty}(1-\frac{2}{z_n})(1-\frac{1}{z_n})^{-2} = \pi\prod_{n=1}^{\infty}(1-\frac{2}{z_n})\frac{z_n^2}{(z_n-1)^2} = \pi\prod_{n=1}^{\infty}\frac{z_n}{z_n-1}\frac{z_n-2}{z_n-1}.$$

The result now follows since $\zeta(2) = \frac{\pi^2}{6}$.

Remark 4.15. *A similarity of the products in the previous corollaries is worth spelling out here. That is, rewriting Wallis Product as $\frac{\pi}{2} = \frac{-2}{-3}\frac{-2}{-3}\frac{-4}{-5}\frac{-4}{-5}\ldots$ we can now highlight the double occurrence of the trivial zeros of the zeta function in the numerators; in the second product $\frac{\pi}{6} = \frac{z_1}{z_1-1}\frac{z_1-2}{z_1-1}\frac{z_2}{z_2-1}\frac{z_2-2}{z_2-1}\ldots$ the non-trivial zeros of the zeta function appear as well in the numerators (despite the little discrepancy present in the shift by 2 in the second occurrence). The interesting thing, however, is what is present in the denominators of each product.*

We now kind of change direction to the zeta function at odd integers ≥ 3 none of whose exact values is known (unlike the ones at even integers ≥ 2 due to Euler which we mentioned in the introduction) and we specialize by considering the zeta function at 3. Thus the exact value that $\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges to is one of the most notorious problems that did not even yield to Euler. A less difficult problem to consider is to find a representation of $\zeta(3)$ in terms of

$$\sum_{n=1}^{\infty}\frac{1}{n^3\binom{2n}{n}}$$

only which would be a beautiful result similar to Euler's

$$\zeta(2) = 3\sum_{n=1}^{\infty} \frac{1}{n^2 \begin{pmatrix} 2n \\ n \end{pmatrix}}$$

not to mention the fact that it is faster converging.
First for some background:
This author [12] has previously proved that:

$$\zeta(3) = -\frac{\sqrt{3}}{18}\pi^3 + \frac{3\sqrt{3}}{4}\pi \sum_0^{\infty} \frac{1}{(3n+1)^2} - \frac{3}{4}\sum_1^{\infty} \frac{1}{n^3 \binom{2n}{n}}.$$

At that point it was natural to hope to express $\sum_0^{\infty} \frac{1}{(3n+1)^2}$ in terms of π^2 only, given the shift by 1 of an already known series, which would give an amazingly pretty result. However, the exact value of $\sum_1^{\infty} \frac{1}{(3n+1)^2}$ turned out to be a difficult open problem and indeed the series $\sum_0^{\infty} \frac{1}{(n+\frac{1}{3})^2}$ was labelled $\psi^{(1)}(\frac{1}{3})$ in the literature.
We now refine the above discovery to an Euler-type identity that involves the Barnes function (recall that Barnes was the same person who suggested that the Riemann Hypothesis be handled by Littlewood)

Definition 4.16. *The **Barnes-G function** is defined by the functional equation*

$$G(z+1) = \Gamma(z)G(z),\ G(1) = 1.$$

where $\Gamma(z)$ is the Euler gamma function.

Remark 4.17. *Not only does Barnes function generalize the famous gamma function which itself was needed to complete the zeta function but also has an application to the Riemann Hypothesis; Montgomery [24] conjectured that the non-trivial zeros of the zeta function are pairwise distributed like eigenvalues of matrices in the Gaussian Unitary Ensemble of Random Matrix Theory.*

From [1] we can write the following identities:

$$\log\Gamma(\frac{1}{3}) = -\frac{3}{2}\log G(\frac{1}{3}) + \frac{\log 3}{48} + \frac{\pi}{12\sqrt{3}} - 2\log A - \frac{\psi^{(1)}(\frac{1}{3})}{8\pi\sqrt{3}} + \frac{1}{6},$$

$$\log\Gamma(\frac{2}{3}) = -3\log G(\frac{2}{3}) + \frac{\log 3}{24} + \frac{\pi}{6\sqrt{3}} - 4\log A - \frac{\psi^{(1)}(\frac{2}{3})}{4\pi\sqrt{3}} + \frac{1}{3},$$

where A is Glaisher's Constant.
Moreover, $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{2}{3}\sqrt{3}\pi$ using the well-known formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$.
Combining we get:

$$\log(2\pi) - \frac{1}{2}\log 3 =$$

$$-\frac{3}{2}\log G(\frac{1}{3}) - 3\log G(\frac{2}{3}) + \frac{\log 3}{16} + \frac{\pi}{4\sqrt{3}} - 6\log A - \frac{\psi^{(1)}(\frac{1}{3}) + 2\psi^{(1)}(\frac{2}{3})}{8\pi\sqrt{3}} + \frac{1}{2}.$$

Therefore,

$$\psi^{(1)}(\frac{1}{3})+2\psi^{(1)}(\frac{2}{3}) = 4\sqrt{3}\pi\{-2\log(2\pi)+\frac{9}{8}\log 3-6\log(\sqrt{G(\frac{1}{3})}G(\frac{2}{3})-12\log A+1\}+2\pi^2.$$

That is,

$$\sum_0^\infty \frac{1}{(n+\frac{1}{3})^2} + 2\sum_{-\infty}^{-1} \frac{1}{(n+\frac{1}{3})^2} =$$

$$4\sqrt{3}\pi\{-2\log(2\pi) + \frac{9}{8}\log 3 - 6\log(\sqrt{G(\frac{1}{3})}G(\frac{2}{3})) - 12\log A + 1\} + 2\pi^2.$$

Together with

$$\sum_0^\infty \frac{1}{(n+\frac{1}{3})^2} + \sum_{-\infty}^{-1} \frac{1}{(n+\frac{1}{3})^2} = \frac{4\pi^2}{3},$$

we now have

$$\sum_0^\infty \frac{1}{(n+\frac{1}{3})^2} = 4\sqrt{3}\pi\{2\log(2\pi)-\frac{9}{8}\log 3+6\log(\sqrt{G(\frac{1}{3})}G(\frac{2}{3}))+12\log A-1\}+\frac{2}{3}\pi^2.$$

Substituting in the earlier $\zeta(3)$ identity we get

$$\zeta(3) = \pi^2\{2\log(2\pi)-\frac{9}{8}\log 3+6\log(\sqrt{G(\frac{1}{3})}G(\frac{2}{3}))+12\log A-1\}-\frac{3}{4}\sum_1^\infty \frac{1}{n^3\binom{2n}{n}}.$$

Luckily Barnes identity ([3], section 7, p. 288) allows us to write

$$12\log A - 1 = \frac{\log 2}{3} - 2\log\pi - 8\log G(\frac{1}{2})$$

which leads to

$$\zeta(3) = \pi^2\{\frac{7}{3}\log 2 - \frac{9}{8}\log 3 + \log\frac{(G(\frac{1}{3}))^3(G(\frac{2}{3}))^6}{(G(\frac{1}{2}))^8}\} - \frac{3}{4}\sum_1^\infty \frac{1}{n^3\binom{2n}{n}}$$

or

$$\zeta(3) = \frac{7}{3}\pi^2 \log 2 - \frac{9}{8}\pi^2 \log 3 + \pi^2 \log \frac{(G(\frac{1}{3}))^3 (G(\frac{2}{3}))^6}{(G(\frac{1}{2}))^8} - \frac{3}{4}\sum_{1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}$$

or

$$\zeta(3) = \pi^2 \{\frac{7}{3} \log 2 - \frac{9}{8} \log 3 + \log \frac{(G(\frac{1}{3}))^3 (G(\frac{2}{3}))^6}{(G(\frac{1}{2}))^8}\} - \frac{3}{4}\sum_{1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}$$

If we take other constants with the Barnes function, we can obtain a similar alternative representation for $\zeta(3)$ as follows:
In [13] this author obtained the following representation:

$$\zeta(3) = \frac{\pi}{2}\sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{3}}{n^2} - \frac{3}{4}\sum_{1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}.$$

Now in [2] Adamchik states the reflection formula for the Barnes function:

$$\log \frac{G(1+z)}{G(1-z)} = -z \log \frac{\sin(\pi z)}{\pi} - \frac{1}{2\pi} Cl_2(2\pi z),$$

for $0 < z < 1$, where $Cl_2(z) = \sum_{n=1}^{\infty} \frac{\sin(nz)}{n^2}$.
Upon taking $z = \frac{1}{6}$ we obtain

$$\zeta(3) = \pi^2 \{\frac{1}{6} \log 2\pi - \log \frac{G(\frac{7}{6})}{G(\frac{5}{6})}\} - \frac{3}{4}\sum_{1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}$$

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