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Contributions to the Theory of the Barnes Function

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Abstract

This paper presents a family of new integral representations and asymptotic series of the multiple gamma function. The numerical schemes for high-precision computation of the Barnes gamma function and Glaisher's constant are also discussed.

1 Introduction

In a sequence of papers published between 1899-1904, Barnes introduced and studied (see [9, 10, 11, 12]) a generalization of the classical Euler gamma function, called the multiple gamma function $\Gamma_n(z)$. The function $\Gamma_n(z)$ satisfies the following recurrence-functional equation [31, 32]:

$$\begin{aligned} &\Gamma_{n+1}(z+1) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)}, \quad z \in \mathbb{C}, \quad n \in \mathbb{N}, \\ &\Gamma_1(z) = \Gamma(z), \\ &\Gamma_n(1) = 1, \end{aligned} \tag{1.1}$$

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where $\Gamma(z)$ is the Euler gamma function. The system (1.1) has a unique solution if we add the condition of convexity [32]

$$(-1)^{n+1}\frac{d^{n+1}}{d\,z^{n+1}}\,\log\Gamma_n(z)\geq 0,\quad z>0.$$

In this paper we aim at the most interesting case $G(z)=1/\Gamma_2(z)$, being the so-called double gamma function or the Barnes G-function:

$$\begin{aligned}&G(z+1)=\Gamma(z)\,G(z),\quad z\in\mathbb{C},\\&G(1)=G(2)=G(3)=1,\\&\frac{d^3}{d\,z^3}\,\log G(z)\geq 0,\quad z>0.\end{aligned}$$

The G-function has several equivalent forms including the Weierstrass canonical product

$$G(z+1)=(2\pi)^{\frac{z}{2}}\exp\Big(-\frac{z+z^2(1+\gamma)}{2}\Big)\prod_{k=1}^{\infty}\Big(1+\frac{z}{k}\Big)^k\exp\Big(\frac{z^2}{2k}-z\Big)$$

and the functional relationship with the Hurwitz zeta function $\zeta(t,z)$, due to Gosper [20] and Vardi [31]

$$\log G(z+1)-z\,\log\Gamma(z)=\zeta'(-1)-\zeta'(-1,z),\quad \Re(z)>0,\tag{1.3}$$

where γ denotes the Euler-Mascheroni constant and the derivative $\zeta'(t,z)=\frac{d}{dt}\zeta(t,z)$ is defined as an analytic continuation provided by the Hermite integral (3.19):

$$\begin{aligned}&\zeta'(-1,z)=\frac{z^2}{2}\log z-\frac{z^2}{4}-\frac{z}{2}\log z+\\&2\,z\int_0^\infty\frac{\arctan(\frac{x}{z})}{e^{2\pi x}-1}\,dx+\int_0^\infty\frac{x\,\log(x^2+z^2)}{e^{2\pi x}-1}\,dx.\quad \Re(z)>0,\end{aligned}\tag{1.4}$$

In particular,

$$\zeta'(-1)=\zeta'(-1,0)=2\int_0^\infty\frac{x\,\log x}{e^{2\pi x}-1}\,dx.\tag{1.5}$$

For $z \to \infty$, the G-function has the Stirling asymptotic expansion

$$\log G(z+1) = \frac{z}{2}\log(2\pi) - \log A - \frac{3z^2}{4} + \left(\frac{z^2}{2} - \frac{1}{12}\right)\log z + O\Big(\frac{1}{z}\Big),$$

where A is the Glaisher-Kinkelin constant given by

$$\log A = \frac{1}{12} - \zeta'(-1). \tag{1.6}$$

Combining this definition with the representation (1.5), we obtain a new integral for $\log A$:

$$\log A = \frac{1+\log(2\pi)}{12} - \frac{1}{2\pi^2}\int_0^\infty \frac{x\log x}{e^x - 1}\,dx. \tag{1.7}$$

The constant A originally appeared in papers by Kinkelin [23] and Glaisher [17, 18, 19] on the asymptotic expansion (when $n \to \infty$) of the following product

$$1^{1^p}\, 2^{2^p} \ldots n^{n^p}, \quad p \in \mathbb{N}.$$

Similar to the Euler gamma function, the G-function satisfies the multiplication formula [31]:

$$\begin{aligned} G(n\,z) &= e^{\zeta'(-1)(1-n^2)} n^{n^2 z^2/2 - nz + 5/12} \\ &(2\pi)^{\frac{n-1}{2}(1-nz)} \prod_{i=0}^{n-1}\prod_{j=0}^{n-1} G\Big(z + \frac{i+j}{n}\Big), \quad n \in \mathbb{N} \end{aligned} \tag{1.8}$$

The Barnes function is not listed in the tables of the most well-known special functions. However, it is cited in the exercises by Whittaker and Watson [33], and in entries 6.441, 8.333 by Gradshteyn and Ryzhik [21]. Recently, Vardi [31], Quine and Choi [27], and Kumagai [24] computed the functional determinants of Laplacians of the n-sphere in terms of the Γ_n-function. Choi, Srivastava [13, 14] and Adamchik [3, 16] presented relationships (including integrals and series) between other mathematical functions and the multiple gamma function.

The Γ_n function has several interesting applications in pure and applied mathematics and theoretical physics. The most intriguing one is the application of G-function to the Riemann Hypothesis. Montgomery [26] and Sarnak [29] (see [22] for additional references) have conjectured that the non-trivial zeros of the Riemann zeta function are pairwise distributed like eigenvalues of matrices in the Gaussian Unitary Ensemble (GUE) of random matrix theory. It has been shown in works by Mehta, Sarnak, Conrey, Keating, and Snaith that a closed representation for statistical averages over GUE of $N \times N$ unitary matrices, when $N \to \infty$ can be expressed in terms of the Barnes functions.

Another interesting appearance of the G-function is in its relation to the determinants of the Hankel matrices. Consider a matrix M of the Bell numbers:

$$M_n = \begin{pmatrix} B_1 & B_2 & \dots & B_n \\ B_2 & B_3 & \dots & B_{n+1} \\ & & \dots & \\ B_n & B_{n+1} & \dots & B_{2n-1} \end{pmatrix}$$

where B_n are the Bell numbers, defined by

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$$

and $\left\{ {n \atop k} \right\}$ are the Stirling subset numbers [28]

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}, \quad \left\{ {n \atop 0} \right\} = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0 \end{cases} \tag{1.9}$$

It was proved in [5], that

$$\det M_n = G(n+1), \quad n \in \mathbb{N}.$$

2 Special cases of the Barnes Function

For some particular arguments, the Barnes function can be expressed in terms of the known special functions and constants. The simplest (but not-trivial) special case is due to Barnes [9]:

$$\log \mathrm{G}\left(\frac{1}{2}\right) = \frac{1}{8} + \frac{\log 2}{24} - \frac{\log \pi}{4} - \frac{3}{2} \log A, \tag{2.10}$$

where A is the Glaisher-Kinkelin constant (1.6). The next two special cases are due to Srivastava and Choi [14]:

$$\log G\Big(\frac{1}{4}\Big) = \frac{3}{32} - \frac{\mathbf{G}}{4\pi} - \frac{3}{4}\log\Gamma\Big(\frac{1}{4}\Big) - \frac{9}{8}\log A, \tag{2.11}$$

$$\log G\Big(\frac{3}{4}\Big) = \log \mathrm{G}\Big(\frac{1}{4}\Big) + \frac{\mathbf{G}}{2\pi} - \frac{\log 2}{8} - \frac{\log\pi}{4} + \log\Gamma\Big(\frac{1}{4}\Big), \tag{2.12}$$

where $\mathbf{G}$ denotes Catalan's constant:

$$\mathbf{G} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}.$$

Identities (2.11) and (2.12) are proven by using the duplication formula (1.8) (setting $n = 2$) together with (2.10). More special cases for $z = \frac{1}{6}, \frac{1}{3}, \frac{2}{3}$ and $\frac{5}{6}$ can be derived as well. Here is one of such formulas (see [3] for the others):

Proposition 2.1.

$$\begin{aligned}\log G\left(\frac{1}{3}\right) =& \frac{1}{9} + \frac{\log 3}{72} + \frac{\pi}{18\sqrt{3}} - \\ & \frac{2}{3}\log\Gamma\left(\frac{1}{3}\right) - \frac{4}{3}\log A - \frac{\psi^{(1)}\left(\frac{1}{3}\right)}{12\sqrt{3}\pi},\end{aligned} \tag{2.13}$$

where $\psi^{(1)}(z) = \frac{\partial^2}{\partial z^2}\log\Gamma(z)$ *is the polygamma function.*

Proof. We recall the Lerch functional equation for the Hurwitz zeta function (see [2, 8])

$$\begin{aligned}&\zeta'(-n,z) + (-1)^n\zeta'(-n,1-z) = \frac{\pi i}{n+1}\,B_{n+1}(z) + \\ &e^{-\pi i n/2}\frac{n!}{(2\pi)^n}\,\mathrm{Li}_{n+1}\big(e^{2\pi i z}\big), \quad 0 < z < 1, \quad n \in \mathbb{N},\end{aligned}$$

where $\zeta'(t,z) = \frac{d}{dt}\zeta(t,z)$, $B_n(z)$ are the Bernoulli polynomials and $\mathrm{Li}_n(z)$ is the polylogarithm. Setting $n = 1$ and using (1.3) yields the reflexion formula for the G-function (see also [14, 16]):

$$\log\left(\frac{G(1+z)}{G(1-z)}\right) = -z\log\left(\frac{\sin(\pi z)}{\pi}\right) - \frac{1}{2\pi}\mathrm{Cl}_2(2\,\pi z), \quad 0 < z < 1, \tag{2.14}$$

where $\mathrm{Cl}_2(z)$ is the Clausen function defined by

$$\mathrm{Cl}_2(z) = -\Im(\mathrm{Li}_2(e^{-iz})). \tag{2.15}$$

Upon setting $z = \frac{1}{3}$ to (2.14), and making use of

$$\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) = \frac{\psi^{(1)}\left(\frac{1}{3}\right)}{3\sqrt{3}} - \frac{2\pi^2}{9\sqrt{3}},$$

we obtain

$$\frac{G\left(\frac{1}{3}\right)}{G\left(\frac{2}{3}\right)} = \frac{\sqrt[3]{2\pi}}{\sqrt[6]{3}\,\Gamma\left(\frac{1}{3}\right)} \exp\left(\frac{2\pi^2 - 3\psi^{(1)}\left(\frac{1}{3}\right)}{18\pi\sqrt{3}}\right). \tag{2.16}$$

On the other hand, using the multiplication formula (1.8) with $z = \frac{1}{3}$ and $n = 3$, we find

$$G\left(\frac{1}{3}\right)\mathrm{G}\left(\frac{2}{3}\right) = \sqrt[3]{\frac{3^{7/12}e^{2/3}}{2\pi A^8\Gamma\left(\frac{1}{3}\right)}}. \tag{2.17}$$

Now, combining (2.16) and (2.17), we conclude the proof. □

It remains to be seen if the Barnes function (or the multiple gamma function) can be expressed as a finite combination of elementary functions and constants for other rational arguments. The most general result available for the multiple gamma function $\Gamma_n(z)$ is a closed form at $z = \frac{1}{2}$:

Proposition 2.2. *For $n \in \mathbb{N}$*

$$\begin{aligned}(-1)^n(n-1)!\log\Gamma_n(\frac{1}{2}) &= -\frac{(2n-3)!!\log\pi}{2^n} + \\ \log 2\sum_{k=1}^{n}\frac{P_{k,n}(\frac{1}{2})\,B_{k+1}}{(k+1)\,2^k} &+ \sum_{k=1}^{n}\frac{2^k-1}{2^k}P_{k,n}(\frac{1}{2})\zeta'(-k),\end{aligned} \tag{2.18}$$

where $P_{k,n}(\frac{1}{2})$ are coefficients by x^k, $k \le n$ in expansion of

$$\prod_{j=1}^{n-1}(x + j - \frac{1}{2})$$

Proof. The proof follows directly from [4], formula (17), by setting $z = \frac{1}{2}$. □

Here are a few particular cases:
the gamma function ($n = 1$):

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

the Barnes function ($n = 2$):

$$G(\frac{1}{2}) = \frac{2^{1/24}\, e^{1/8}}{A^{3/2}\, \pi^{1/4}}$$

the triple gamma function ($n = 3$):

$$\Gamma_3(\frac{1}{2}) = \frac{A^{3/2}\, \pi^{3/16}}{2^{1/24}}\, \exp\left(\frac{7\,\zeta(3)}{32\,\pi^2} - \frac{1}{8}\right)$$

3 Hermite integrals

In this section we derive a few integral representations for the Barnes function. We start with recalling the Hermite integral (see [7, 25]) for the Hurwitz zeta function:

$$\zeta(s,z) = \frac{z^{-s}}{2} + \frac{z^{1-s}}{s-1} + 2\int_0^\infty \frac{\sin(s\arctan(x/z))}{(x^2+z^2)^{s/2}(e^{2\pi x}-1)}\, dx, \tag{3.19}$$

which provides analytic continuation of $\zeta(s,z)$ to the domain $s \in \mathbb{C} - \{1\}$. Differentiating both sides of (3.19) with respect to s, letting $s = -1$ (and $s = -2$), and using the second Binet formula for $\log\Gamma(z)$ (see [7])

$$\log\Gamma(z) = (z - \frac{1}{2})\log z - z + \frac{\log(2\pi)}{2} + 2\int_0^\infty \frac{\arctan(x/z)}{e^{2\pi x}-1}\, dx \tag{3.20}$$

together with (1.3), we readily obtain:

$$\begin{aligned} \log G(z+1) = \frac{z^2}{2}\Big(\log z - H_2\Big) - \Big(z\,\zeta'(0) - \zeta'(-1)\Big) - \\ \int_0^\infty \frac{x\log(x^2+z^2)}{e^{2\pi x}-1}\, dx, \quad \Re(z) > 0, \end{aligned} \tag{3.21}$$

where $H_p = \sum_{k=1}^{p} 1/k$ are the harmonic numbers and $\zeta'(z)$ is a derivative of the Riemann zeta function.

In the same manner, repeating the above steps n times, we derive the general integral representations.

Proposition 3.1. *For $\Re(z) > 0$*

$$\begin{aligned}
&(2n)!\,\log\Gamma_{2n+1}(z+1) = \\
&\quad \frac{z^{2n+1}(\log z - H_{2n+1})}{2n+1} - \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\zeta'(-k)\,z^{2n-k} - \\
&\quad \sum_{k=1}^{2n-1}(-1)^k k!\left\{{2n \atop k}\right\}\log\Gamma_{k+1}(z+1) + \\
&\quad 2\,(-1)^n\int_0^\infty \frac{x^{2n}\arctan\left(x/z\right)}{e^{2\pi x}-1}\,dx
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
(2n-1)!\log\Gamma_{2n}(z+1) = &-\frac{z^{2n}(\log z - H_{2n})}{2n} + \\
&\sum_{k=0}^{2n-1}(-1)^k\binom{2n-1}{k}\zeta'(-k)z^{2n-k-1} + \\
&\sum_{k=1}^{2n-2}(-1)^k k!\left\{{2n-1 \atop k}\right\}\log\Gamma_{k+1}(z+1) - \\
&(-1)^n\int_0^\infty \frac{x^{2n-1}\log(x^2+z^2)}{e^{2\pi x}-1}\,dx
\end{aligned} \tag{3.23}$$

where $\left\{{n \atop k}\right\}$ are the Stirling subset numbers, defined in (1.9).

Upon substituting $z = 1$ into formulas (3.22) and (3.23), we obtain new integrals

$$2\,(-1)^n \int_0^\infty \frac{x^{2n}\arctan(x)}{e^{2\pi x}-1}\,dx = \frac{H_{2n+1}}{2n+1} + \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \zeta'(-k) \qquad (3.24)$$

$$(-1)^n \int_0^\infty \frac{x^{2n-1}\log(1+x^2)}{e^{2\pi x}-1}\,dx = \frac{H_{2n}}{2n} + \sum_{k=0}^{2n-1} (-1)^k \binom{2n-1}{k} \zeta'(-k) \qquad (3.25)$$

Performing simple transformations of the integrands in (3.24) and (3.25) will apparently lead to a large number of special integrals:

$$2\int_0^\infty \frac{x\log x}{e^{2\pi x}-1}\,dx = \zeta'(-1)$$

$$\int_0^\infty \frac{x\log x}{e^{2\pi x}+1}\,dx = 12\zeta'(-1) + \log 2$$

$$\int_0^\infty \frac{x\log(1+x^2)}{e^{2\pi x}+1}\,dx = \frac{3}{4} - \frac{23}{24}\log 2 + \frac{1}{2}\zeta'(-1)$$

$$2\int_0^\infty \frac{\arctan\left(\frac{x}{z}\right)}{e^{2\pi x}+1}\,dx = z\log z - z + \frac{\log(2\pi)}{2} - \log\Gamma\big(z+\frac{1}{2}\big), \qquad \Re(z) > 0$$

$$4\int_0^\infty \frac{\arctan(x)}{e^{2\pi x}+1}\,dx = -2 + 3\log 2$$

$$2\int_0^\infty \frac{x\,dx}{(e^{2\pi x}+1)(x^2+z^2)} = \psi\big(z+\frac{1}{2}\big) - \log z, \qquad \arg(z) \neq \frac{\pi}{2}$$

$$2\int_0^\infty \frac{x\,dx}{(e^{2\pi x}-1)(x^2+z^2)} = \log z - \psi(z) - \frac{1}{2z}, \qquad \arg(z) \neq \frac{\pi}{2}$$

4 Binet-like representation

In this section we derive the Binet integral representation for the Barnes function.

Proposition 4.1. *The Barnes G-function admits the Binet integral representation:*

$$\log G(z+1) = z\log\Gamma(z) + \frac{z^2}{4} - \frac{\log z}{2}B_2(z) - \log A - \int_0^\infty \frac{e^{-zx}}{x^2}\Big(\frac{1}{1-e^{-x}} - \frac{1}{x} - \frac{1}{2} - \frac{x}{12}\Big)\,dx, \quad \Re(z) > 0. \tag{4.26}$$

Proof. Recall the well-known integral for the Hurwitz function [8]

$$\zeta(s,z) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}e^{-zx}}{1-e^{-x}}\,dx, \quad \Re(s) > 1, \quad \Re(z) > 0 \tag{4.27}$$

The integral in the right hand-side of (4.27) can be analytically continued to the larger domain $\Re(s) > -2$ by subtracting the truncated Taylor series of $\frac{1}{1-e^{-x}}$ at $x = 0$. Since

$$\frac{1}{1-e^{-x}} = \frac{1}{x} + \frac{1}{2} + \frac{x}{12} + O(x^3) \tag{4.28}$$

and

$$\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}e^{-zx}\Big(\frac{1}{x} + \frac{1}{2} + \frac{x}{12}\Big)\,dx = \frac{s}{12}z^{-1-s} + \frac{z^{-s}}{2} - \frac{z^{1-s}}{1-s},$$
$$\Re(s) > 1, \Re(z) > 0$$

we therefore can rewrite (4.27) as

$$\zeta(s,z) = -\frac{z^{1-s}}{1-s} + \frac{z^{-s}}{2} + \frac{s}{12}z^{-s-1} + \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}e^{-zx}\Big(\frac{1}{1-e^{-x}} - \frac{1}{x} - \frac{1}{2} - \frac{x}{12}\Big)\,dx, \quad \Re(s) > -2.$$

Differentiating the above formula with respect to s and computing the limit at $s = -1$, we obtain

$$\zeta'(-1,z) = \frac{1}{12} - \frac{z^2}{4} + \frac{\log z}{2} B_2(z) +$$
$$\int_0^\infty \frac{e^{-zx}}{x^2}\Big(\frac{1}{1-e^{-x}} - \frac{1}{x} - \frac{1}{2} - \frac{x}{12}\Big)\,dx, \quad \Re(z) > 0,$$

where $B_2(z)$ is the second Bernoulli polynomial. The convergence of the above integral is ensured by (4.28). Finally, using (1.3), we conclude the proof. □

5 Asymptotic expansion of $\Gamma_n(z)$

The asymptotic expansion for $\Gamma_n(z)$ when $z \to \infty$ follows straightforwardly from formulas (3.22) and (3.23) by expanding $\log(1+x^2)$ and $\arctan(x)$ into the Taylor series, and then performing formal term-by-term integration.

Lemma 5.1. *For* $\Re(z) > 0$

$$\log G(z+1) = \frac{z^2}{2}\left(\log z - \frac{3}{2}\right) - \frac{\log z}{12} - z\,\zeta'(0) + \zeta'(-1) -$$
$$\sum_{k=1}^{n} \frac{B_{2k+2}}{4\,k\,(k+1)\,z^{2k}} + O\!\left(\frac{1}{z^{2n+2}}\right), \quad z \to \infty. \tag{5.29}$$

Proof. We start with the integral representation (3.21). By recalling the Taylor series for $\log(1+x)$

$$\log\!\left(1+\frac{x}{z}\right) = \sum_{k=0}^{n} \frac{(-1)^k}{k+1}\left(\frac{x}{z}\right)^{k+1} - \frac{(-1)^n}{z^{n+1}} \int_0^x \frac{y^{n+1}}{y+z}\,dy,$$

we find that

$$\int_0^\infty \frac{x\log(1+x^2/z^2)}{e^{2\pi x}-1}\,dx =$$
$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1}\frac{1}{z^{2k+2}} \int_0^\infty \frac{x^{2k+3}}{e^{2\pi x}-1}\,dx - \frac{(-1)^n}{z^{2n+2}} \int_0^\infty \frac{t\,dt}{e^{2\pi t}-1} \int_0^{t^2} \frac{y^{n+1}}{y+z^2}\,dy$$

which, by appealing to (formula 1.6.4 in [7])

$$\int_0^\infty \frac{t^{2k-1}}{e^{2\pi t}-1}\,dt = (-1)^{k+1}\,\frac{B_{2k}}{4\,k}, \quad k \in \mathbb{N}, \tag{5.30}$$

reduces to

$$\int_0^\infty \frac{x\log(1+x^2/z^2)}{e^{2\pi x}-1}\,dx =$$
$$-\sum_{k=1}^{n+1}\frac{B_{2k+2}}{4\,k\,(k+1)z^{2k}} - 2\,(-1)^n\int_0^\infty \frac{t\,dt}{e^{2\pi t}-1}\int_0^{t/z}\frac{y^{2n+3}}{1+y^2}\,dy$$

Substituting this into (3.21), we complete the proof. □

Lemma 5.2. *For* $\Re(z) > 0$

$$\begin{aligned}2\log\Gamma_3(z+1) = &-\frac{z^3}{3}\Big(\log z - \frac{11}{6}\Big) + \frac{z^2}{2}\Big(\log z - \frac{3}{2} + \zeta'(0)\Big) - \\ &\frac{\log z}{12} - z\Big(\zeta'(0) + 2\zeta'(-1)\Big) + \zeta'(-1) + \zeta'(-2) - \\ &\sum_{k=1}^{n}\frac{B_{2k+2}}{4\,k\,(k+1)\,z^{2k}} - \sum_{k=1}^{n}\frac{B_{2k+2}}{2\,(k+1)\,(2k-1)\,z^{2k-1}} + \\ &O(\frac{1}{z^{2n+1}}), \quad z \to \infty.\end{aligned} \tag{5.31}$$

Proof. Employing the same technique as in getting (5.29), we begin with the integral representation of the triple gamma function:

$$\begin{aligned}2\log\Gamma_3(z+1) = &-\frac{z^3}{2}(\log z - H_3) + \log G(z+1) + \\ &\Big(z^2\zeta'(0) - 2z\zeta'(-1) + \zeta'(-2)\Big) + 2\int_0^\infty \frac{x^2\arctan(x/z)}{e^{2\pi x}-1}\,dx\end{aligned} \tag{5.32}$$

that follows directly from Proposition 3.1 with $n = 1$. Expanding *arctan* into the Taylor series

$$\arctan(\frac{x}{z}) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} (\frac{x}{z})^{2k-1} + \frac{(-1)^n}{z^{2n-1}} \int_0^x \frac{y^{2n}}{y^2+z^2}\, dy$$

and making use of (5.30), we readily obtain

$$\int_0^\infty \frac{x^2 \arctan(x/z)}{e^{2\pi x}-1}\, dx = -\sum_{k=1}^{n} \frac{B_{2k+2}}{4\,(k+1)(2k-1)\, z^{2k-1}} +$$

$$\frac{(-1)^n}{z^{2n-1}} \int_0^\infty \frac{t^2\, dt}{e^{2\pi t}-1} \int_0^t \frac{y^{2n}}{y^2+z^2}\, dy.$$

The conclusion of the lemma follows immediately by substituting this into formula (5.32) and using Lemma 5.1.

□

6 Implementation

In this section we discuss a numeric computational scheme for the Barnes function. Let us consider an integral representation dated back to Alexejewsky [6] and Barnes [9]:

$$\int_0^z \log\Gamma(x) dx = \frac{z(1-z)}{2} + \frac{z}{2}\log(2\pi) + z\log\Gamma(z) - \log G(z+1),$$

where $\Re(z) > -1$. Employing integration by parts, we derive

$$\log G(z+1) = \frac{z(1-z)}{2} + \frac{z}{2}\log(2\pi) + \int_0^z x\,\psi(x)\, dx, \quad \Re(z) > -1 \quad (6.33)$$

The integral representation (6.33) provides an efficient numeric procedure for arbitrary precision evaluation of the G-function. This representation demonstrates that the complexity of computing $G(z)$ depends at most on the computational complexity of the polygamma function. The later can be numerically computed by the Spouge approximation [30] (a modification of the Lanczos approximation). For the numerical integration in (6.33) we shall use the Gaussian quadrature scheme.

The restriction $\Re(z) > -1$ in (6.33) can be easily removed by analyticity of the polygamma. For example, by resolving the singularity of the integrand at the pole $x = -1$, we continue $\log G(z+1)$ to the wider area $\Re(z) > -2$:

$$\log G(z+1) = \frac{z(1-z)}{2} + \frac{z}{2}\log(2\pi) + \log(z+1) + \int_0^z \left(x\psi(x) - \frac{1}{x+1} \right) dx$$

Another method of the analytic continuation of the G-function is to use the following well-known identity

$$\psi(x) = \psi(-x) - \pi\cot(\pi x) - \frac{1}{x}$$

which, upon substituting it into (6.33), yields

$$\log G(1-z) = \log G(1+z) - z\log(2\pi) + \int_0^z \pi x \cot(\pi x)\, dx. \tag{6.34}$$

The identity (6.34) holds everywhere in a complex plane of z, except the real axes, where the integrand has simple poles. Therefore, combining (6.33) with (6.34) yields the following definition of the Barnes function

$$\begin{aligned} &G(-n) = 0, \quad n \in \mathbb{N}, \\ &G(z) = (2\pi)^{(z-1)/2} \exp\left(-\frac{(z-1)(z-2)}{2} + \int_\gamma x\,\psi(x)\,dx \right), \qquad (6.35) \\ &\qquad \arg(z) \neq \pi, \end{aligned}$$

where the contour of integration γ is a line between 0 and $z-1$ that does not cross the negative real axis; for example, γ could be the following path $\{0, i, i+z-1, z-1\}$.

It remains to define $G(z)$ for $z \in \mathbb{R}^-$. We do this by using the reflexion formula (2.14), which upon periodicity of the Clausen function (2.15) can be rewritten as follows:

$$\begin{aligned} G(-z) = (-1)^{\lfloor z/2 \rfloor - 1} G(z+2) \left| \frac{\sin(\pi z)}{\pi} \right|^{z+1} * \\ \exp\left(\frac{1}{2\pi} \mathrm{Cl}_2\big(2\pi(z - \lfloor z \rfloor)\big) \right), \quad z \in \mathbb{R}^-. \end{aligned} \tag{6.36}$$

7 Glaisher's constant

In this section we discuss numeric computational schemes for Glaisher's constant:

$$A = \exp\left(\frac{1}{12} - \zeta'(-1)\right) \tag{7.37}$$

In light of the functional equation for the zeta function

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s)\zeta(1-s)\sin\left(\frac{\pi s}{2}\right)$$

we can rewrite (7.37) as

$$A = \exp\left(\frac{\gamma}{12} - \frac{\zeta'(2)}{2\pi^2}\right)(2\pi)^{1/12}. \tag{7.38}$$

Exactly this formula is used in *Mathematica* V4.2 for computing Glaisher's constant.

Another computational scheme for Glaisher's constant is based on the Barnes G-function identity:

$$\log A = \frac{1}{12} + \frac{\log 2}{36} - \frac{\log \pi}{6} - \frac{2}{3}\log G\Big(\frac{1}{2}\Big). \tag{7.39}$$

Here $G(\frac{1}{2})$ can be computed via integral (6.33).

A more efficient algorithm is provided by the following identity

$$\log A = \frac{\log 2}{12} + \frac{1}{36}\sum_{k=1}^{\infty}\Big(\zeta(2k+1) - 1\Big)\Big(28 + \frac{3}{1+k} - \frac{6}{2+k}\Big), \tag{7.40}$$

which allows to approximate Glaisher's constant as

$$\log A = \frac{\log 2}{12} + \frac{1}{36}\sum_{k=1}^{N}\Big(\zeta(2k+1) - 1\Big)\Big(28 + \frac{3}{1+k} - \frac{6}{2+k}\Big) + O\Big(\frac{1}{4^N}\Big)$$

This approximation is suitable for arbitrary precision computation - it requires $\lceil \frac{p}{2}\log_2 10\rceil$ terms to achieve p decimal-digit accuracy.

We close this section with a proof of formula (7.40). Recall that [8]

$$\zeta(s) - 1 = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x(e^x-1)}\, dx, \quad \Re(s) > 1. \tag{7.41}$$

Upon replacing $\zeta(2k+1) - 1$ in (7.40) by its integral representation (7.41) and inverting the order of summation and integration, we obtain

$$\log A = \frac{\log 2}{12} + \frac{1}{36} \int_0^\infty \frac{dx}{e^x(e^x-1)} \sum_{k=1}^\infty \frac{t^{2k}}{(2k)!} \left(28 + \frac{3}{1+k} - \frac{6}{2+k}\right). \tag{7.42}$$

Next, noting that

$$\sum_{k=1}^\infty \frac{t^{2k}}{(2k)!} = \cosh(t) - 1$$

and

$$\sum_{k=1}^\infty \frac{t^{2k}}{(2k)!\,(k+s)} = \frac{2}{t^{2s}} \int_0^t x^{2s-1}(\cosh(x) - 1)\, dx,$$

we compute the inner sum

$$\frac{t^4}{2} \sum_{k=1}^\infty \frac{t^{2k}}{(2k)!} \left(28 + \frac{3}{1+k} - \frac{6}{2+k}\right) = -36 + 3\,t^2 - 14\,t^4 +$$
$$\left(36 + 15\,t^2 + 14\,t^4\right) \cosh(t) - \left(36\,t + 3\,t^3\right) \sinh(t)$$

In the next step, we substitute this back to (7.42) and, employed by (4.27), evaluate the integral. Unfortunately, integration cannot be done term by term. The way around is to multiply the integrand by t^λ, $\lambda > 0$ and consider the limit as $\lambda \to 0$. This yields

$$\int_0^\infty \left(\frac{36\,t^{\lambda-4}}{e^t-1} + \frac{36\,t^{\lambda-4}}{e^{2\,t}\,(e^t-1)} - \frac{72\,t^{\lambda-4}}{e^t\,(e^t-1)} - \frac{36\,t^{\lambda-3}}{e^t-1} + \right.$$

$$\frac{36\,t^{\lambda-3}}{e^{2\,t}\,(e^t-1)} + \frac{15\,t^{\lambda-2}}{e^t-1} + \frac{15\,t^{\lambda-2}}{e^{2\,t}\,(e^t-1)} + \frac{6\,t^{\lambda-2}}{e^t\,(e^t-1)} - \frac{3\,t^{\lambda-1}}{e^t-1} +$$

$$\left. \frac{3\,t^{\lambda-1}}{e^{2\,t}\,(e^t-1)} + \frac{14\,t^{\lambda}}{e^t-1} + \frac{14\,t^{\lambda}}{e^{2\,t}\,(e^t-1)} - \frac{28\,t^{\lambda}}{e^t\,(e^t-1)} \right) dt =$$

$$36\,\Gamma(\lambda-3) - 9\,2^{5-\lambda}\,\Gamma(\lambda-3) - 36\,\Gamma(\lambda-2) - 9\,2^{4-\lambda}\,\Gamma(\lambda-2) -$$

$$21\,\Gamma(\lambda-1) - 15\,2^{1-\lambda}\,\Gamma(\lambda-1) - 3\,\Gamma(\lambda) - \frac{3\,\Gamma(\lambda)}{2^\lambda} + 14\,\Gamma(\lambda+1) -$$

$$\frac{7\,\Gamma(\lambda+1)}{2^\lambda} + 36\,\Gamma(\lambda-1)\,\zeta(\lambda-1).$$

Computing the limit as $\lambda \to 0$, we find that the right hand-side of the above expression evaluates to

$$3 - 3\log 2 - 36\,\zeta'(-1).$$

This completes the proof of (7.40).

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