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An Extension of the Abundancy Index to Certain Quadratic Rings

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Abstract

We begin by introducing an extension of the traditional abundancy index to imaginary quadratic rings with unique factorization. After showing that many of the properties of the traditional abundancy index continue to hold in our extended form, we investigate what we call n-powerfully solitary numbers in these rings. This definition serves to extend the concept of solitary numbers, which have been defined and studied in the integers. We end with some open questions and a conjecture.

1 Introduction

The arithmetic functions σ_k are defined, for every integer k, by $\sigma_k(n) = \sum_{\substack{c|n \\ c>0}} c^k$, and it is conventional to write $\sigma_1 = \sigma$. It is well-known that, for each integer $k \neq 0$, σ_k is multiplicative and satisfies $\sigma_k(p^\alpha) = \dfrac{p^{k(\alpha+1)} - 1}{p^k - 1}$ for all (integer) primes p and positive integers α. The abundancy index of a positive integer n is defined by $I(n) = \dfrac{\sigma(n)}{n}$. Using the formulas $\sigma(n) = \prod_{p^\alpha \| n} \dfrac{p^{\alpha+1} - 1}{p - 1}$

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and $\sigma_{-1}(n) = \prod_{p^\alpha \| n} \frac{p^{-(\alpha+1)} - 1}{p^{-1} - 1}$, it is easy to see that $I = \sigma_{-1}$. Some of the most common questions associated with the abundancy index are those related to friendly numbers.

Two or more distinct positive integers are said to be friends (with each other) if they have the same abundancy index. For example, $I(6) = I(28) = I(496) = 2$, so 6, 28, and 496 are friends. A positive integer that has at least one friend is said to be friendly, and a positive integer that has no friends is said to be solitary. Clearly, 1 is solitary as $I(n) > 1 = I(1)$ for any positive integer $n > 1$. It is also not difficult to show, using the fact that $I = \sigma_{-1}$, that every prime power is solitary. In the next section, we extend the notions of the abundancy index and friendliness to imaginary quadratic integer rings that are also unique factorization domains. Observing the infinitude of possible such generalizations, we note four important properties of the traditional abundancy index that we wish to preserve (possibly with slight modifications).

- The range of the function I is a subset of the interval $[1, \infty)$.
- If n_1 and n_2 are relatively prime positive integers, then $I(n_1 n_2) = I(n_1) I(n_2)$.
- If n_1 and n_2 are positive integers such that $n_1 | n_2$, then $I(n_1) \leq I(n_2)$, with equality if and only if $n_1 = n_2$.
- All prime powers are solitary.

For any square-free integer d, let $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ be the quadratic integer ring given by

$$\mathcal{O}_{\mathbb{Q}(\sqrt{d})} = \begin{cases} \mathbb{Z}[\frac{1+\sqrt{d}}{2}], & \text{if } d \equiv 1 \mod 4; \\ \mathbb{Z}[\sqrt{d}], & \text{if } d \equiv 2, 3 \mod 4. \end{cases}$$

Throughout the remainder of this paper, we will work in the rings $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ for different specific or arbitrary values of d. We will use the symbol "$|$" to mean "divides" in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ in which we are working. Whenever we are working in a ring other than $\mathbb{Z}$, we will make sure to emphasize when we wish to state that one integer divides another in $\mathbb{Z}$. For example, if we are working in $\mathbb{Z}[i]$, the ring of Gaussian integers, we might say that $1 + i | 1 + 3i$ and that $2 | 6$ in $\mathbb{Z}$. We will also refer to primes in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ as "primes," whereas we will refer to (positive) primes in $\mathbb{Z}$ as "integer primes." Furthermore, we will henceforth focus exclusively on values of d for which $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is a unique factorization domain and $d < 0$. In other words, $d \in K$, where we will define K to be the set $\{-163, -67, -43, -19, -11, -7, -3, -2, -1\}$. The set K is known to be the complete set of negative values of d for which $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is a unique factorization domain [3].

For now, let us work in a ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ such that $d \in K$. We will assume familiarity with Keith Conrad's online notes [1]. For $x, y \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, we say that x and y are associated, denoted $x \sim y$, if and only if $x = uy$ for some unit u in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. Furthermore, we will make repeated use of the following well-known facts.

Fact 1.1. *Let $d \in K$. If p is an integer prime, then exactly one of the following is true.*

- *p is also a prime in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. In this case, we say that p is inert in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.*
- *$p \sim \pi^2$ and $\pi \sim \overline{\pi}$ for some prime $\pi \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. In this case, we say p ramifies (or p is ramified) in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.*
- *$p = \pi\overline{\pi}$ and $\pi \not\sim \overline{\pi}$ for some prime $\pi \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. In this case, we say p splits (or p is split) in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.*

Fact 1.2. *Let $d \in K$. If $\pi \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is a prime, then exactly one of the following is true.*

- *$\pi \sim q$ and $N(\pi) = q^2$ for some inert integer prime q.*
- *$\pi \sim \overline{\pi}$ and $N(\pi) = p$ for some ramified integer prime p.*
- *$\pi \not\sim \overline{\pi}$ and $N(\pi) = N(\overline{\pi}) = p$ for some split integer prime p.*

Fact 1.3. *Let $\mathcal{O}^*_{\mathbb{Q}(\sqrt{d})}$ be the set of units in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. Then $\mathcal{O}^*_{\mathbb{Q}(\sqrt{-1})} = \{\pm 1, \pm i\}$, $\mathcal{O}^*_{\mathbb{Q}(\sqrt{-3})} = \left\{\pm 1, \pm\dfrac{1+\sqrt{-3}}{2}, \pm\dfrac{1-\sqrt{-3}}{2}\right\}$, and $\mathcal{O}^*_{\mathbb{Q}(\sqrt{d})} = \{\pm 1\}$ whenever $d \in K\backslash\{-1, -3\}$.*

2 The Extension of the Abundancy Index

For a nonzero complex number z, let $\arg(z)$ denote the argument, or angle, of z. We convene to write $\arg(z) \in [0, 2\pi)$ for all $z \in \mathbb{C}$. For each $d \in K$, we define the set $A(d)$ by

$$A(d) = \begin{cases} \{z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \backslash \{0\} : 0 \leq \arg(z) < \frac{\pi}{2}\}, & \text{if } d = -1; \\ \{z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \backslash \{0\} : 0 \leq \arg(z) < \frac{\pi}{3}\}, & \text{if } d = -3; \\ \{z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \backslash \{0\} : 0 \leq \arg(z) < \pi\}, & \text{otherwise.} \end{cases}$$

Thus, every nonzero element of $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ can be written uniquely as a unit times a product of primes in $A(d)$. Also, every $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \backslash \{0\}$ is associated to a unique element, which we will call $B(z)$, of $A(d)$. We are now ready to define analogues of the arithmetic functions σ_k.

Definition 2.1. Let $d \in K$, and let $n \in \mathbb{Z}$. Define the function $\delta_n \colon \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \backslash \{0\} \to [1, \infty)$ by

$$\delta_n(z) = \sum_{\substack{x \mid z \\ x \in A(d)}} |x|^n.$$

Remark 2.1. We note that, for each x in the summation in the above definition, we may cavalierly replace x with one of its associates. This is because associated numbers have the same absolute value. In other words, the only reason for the criterion $x \in A(d)$ in the summation that appears in Definition 2.1 is to forbid us from counting associated divisors as distinct terms in the summation, but we may choose to use any of the associated divisors as long as we only choose one. This should not be confused with how we count conjugate divisors (we treat $2+i$ and $2-i$ as distinct divisors of 5 in $\mathbb{Z}[i]$ because $2+i \not\sim 2-i$).

Remark 2.2. We note that, by choosing different values of d, the functions δ_n change dramatically. For example, $\delta_2(3) = 10$ when we work in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$, but $\delta_2(3) = 16$ when we work in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$. Perhaps it would be more precise to write $\delta_n(z, d)$, but we will omit the latter component for convenience. We note that we will also use this convention with functions such as I_n (which we will define soon).

We will say that a function $f \colon \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \backslash \{0\} \to \mathbb{R}$ is multiplicative if $f(xy) = f(x)f(y)$ whenever x and y are relatively prime (have no nonunit common divisors).

Theorem 2.1. *Let $d \in K$, and let $f, g \colon \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \backslash \{0\} \to \mathbb{R}$ be multiplicative functions such that $f(u) = g(u) = 1$ for all units $u \in \mathcal{O}^*_{\mathbb{Q}(\sqrt{d})}$. Define*

$F\colon \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\} \to \mathbb{R}$ *by*

$$F(z) = \sum_{\substack{x,y\in A(d)\\ xy\sim z}} f(x)g(y).$$

Then F is multiplicative.

Proof. Suppose $z_1, z_2 \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$ and $\gcd(z_1, z_2) = 1$. For any $x, y \in A(d)$ satisfying $xy \sim z_1z_2$, we may write $x = x_1x_2$, $y = y_1y_2$ so that $x_1y_1 \sim z_1$ and $x_2y_2 \sim z_2$. To make the choice of x_1, x_2, y_1, y_2 unique, we require $x_1, y_1 \in A(d)$. Conversely, if we choose $x_1, x_2, y_1, y_2 \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$ such that $x_1, y_1 \in A(d)$, $x_1y_1 \sim z_1$, $x_2y_2 \sim z_2$, and $x_1x_2, y_1y_2 \in A(d)$, then we may write $x = x_1x_2$ and $y = y_1y_2$ so that $xy \sim z_1z_2$. To simplify notation, write $B(x_2) = x_3$, $B(y_2) = y_3$, and let C be the set of all ordered quadruples (x_1, x_2, y_1, y_2) such that $x_1, x_2, y_1, y_2 \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$, $x_1, y_1 \in A(d)$, $x_1y_1 \sim z_1$, $x_2y_2 \sim z_2$, and $x_1x_2, y_1y_2 \in A(d)$. We have established a bijection between C and the set of ordered pairs (x, y) satisfying $x, y \in A(d)$ and $xy \sim z_1z_2$. Therefore,

$$F(z_1z_2) = \sum_{\substack{x,y\in A(d)\\ xy\sim z_1z_2}} f(x)g(y) = \sum_{(x_1,x_2,y_1,y_2)\in C} f(x_1x_2)g(y_1y_2)$$

$$= \sum_{(x_1,x_2,y_1,y_2)\in C} f(x_1)f(x_2)g(y_1)g(y_2)$$

$$= \sum_{(x_1,x_2,y_1,y_2)\in C} f(x_1)f(B(x_2))g(y_1)g(B(y_2))$$

$$= \sum_{\substack{x_1,y_1\in A(d)\\ x_1y_1\sim z_1}} f(x_1)g(y_1) \sum_{\substack{x_3,y_3\in A(d)\\ x_3y_3\sim z_2}} f(x_3)g(y_3) = F(z_1)F(z_2).$$

□

Corollary 2.1. *For any integer n, δ_n is multiplicative.*

Proof. Noting that $\delta_n(w_1) = \delta_n(w_2)$ whenever $w_1 \sim w_2$, we may let $f, g\colon \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\} \to \mathbb{R}$ be the functions defined by $f(z) = |z|^n$ and $g(z) = 1$ for all $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$. Then the desired result follows immediately from Theorem 2.1. □

Definition 2.2. For each positive integer n, define the function $I_n\colon \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\} \to [1,\infty)$ by $I_n(z) = \dfrac{\delta_n(z)}{|z|^n}$. We say that two or more numbers $z_1, z_2, \ldots, z_r \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$ are *n-powerfully friendly* (or *n-powerful friends*) *in* $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ if $I_n(z_j) = I_n(z_k)$ and $|z_j| \neq |z_k|$ for all distinct $j,k \in \{1,2,\ldots,r\}$. Any $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$ that has no n-powerful friends in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is said to be *n-powerfully solitary in* $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.

Remark 2.3. Whenever $n = 1$, we will omit the adjective "1-powerfully" in the preceding definitions.

As an example, we will let $d = -1$ so that $\mathcal{O}_{\mathbb{Q}(\sqrt{d})} = \mathbb{Z}[i]$. Let us compute $I_2(9+3i)$. We have $9+3i = 3(1+i)(2-i)$, so $\delta_2(9+3i) = N(1) + N(3) + N(1+i) + N(2-i) + N(3(1+i)) + N(3(2-i)) + N((1+i)(2-i)) + N(3(1+i)(2-i)) = 1+9+2+5+18+45+10+90 = 180$. Then $I_2(9+3i) = \dfrac{180}{N(3(1+i)(2-i))} = 2$. Although $I_2(3+9i)$ is also equal to 2, $3+9i$ and $9+3i$ are not 2-powerful friends in $\mathbb{Z}[i]$ because $|3+9i| = |9+3i|$. We now establish some important properties of the functions I_n.

Theorem 2.2. *Let $n\in\mathbb{N}$, $d\in K$, and $z_1, z_2, \pi \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$ with π a prime. Then, if we are working in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, the following statements are true.*

(a) The range of I_n is a subset of the interval $[1,\infty)$, and $I_n(z_1) = 1$ if and only if z_1 is a unit in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. If n is even, then $I_n(z_1) \in \mathbb{Q}$.

(b) I_n is multiplicative.

(c) $I_n(z_1) = \delta_{-n}(z_1)$.

(d) If $z_1|z_2$, then $I_n(z_1) \leq I_n(z_2)$, with equality if and only if $z_1 \sim z_2$.

(e) If $z_1 \sim \pi^k$ for a nonnegative integer k, then z_1 is n-powerfully solitary in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.

Proof. The first sentence in part (a) is fairly clear, and the second sentence becomes equally clear if one uses the fact that $|z_1|^n\in\mathbb{N}$ whenever n is even. To prove part (b), suppose that z_1 and z_2 are relatively prime elements of $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. Then, by Corollary 2.1, $I_n(z_1z_2) = \dfrac{\delta_n(z_1z_2)}{|z_1z_2|^n} = \dfrac{\delta_n(z_1)\delta_n(z_2)}{|z_1|^n|z_2|^n} = I_n(z_1)I_n(z_2)$. In order to prove part (c), it suffices, due to the truth of part (b), to prove that $I_n(\pi^\alpha) = \delta_{-n}(\pi^\alpha)$ for any prime π and nonnegative integer α. To do so is fairly routine, as

$$I_n(\pi^\alpha) = \frac{\delta_n(\pi^\alpha)}{|\pi^\alpha|^n} = \frac{\sum_{j=0}^{\alpha}|\pi^j|^n}{|\pi^\alpha|^n} = \sum_{j=0}^{\alpha}|\pi^{j-\alpha}|^n$$

$$= \sum_{j=0}^{\alpha}|\pi^{\alpha-j}|^{-n} = \sum_{l=0}^{\alpha}|\pi^l|^{-n} = \delta_{-n}(\pi^\alpha).$$

The truth of statement (d) follows from part (c) because, if $z_1|z_2$, then

$$I_n(z_2) = \delta_{-n}(z_2) = \sum_{\substack{x|z_2 \\ x\in A(d)}} |x|^{-n}$$

$$= \sum_{\substack{x|z_1 \\ x\in A(d)}} |x|^{-n} + \sum_{\substack{x|z_2 \\ x\nmid z_1 \\ x\in A(d)}} |x|^{-n} = I_n(z_1) + \sum_{\substack{x|z_2 \\ x\nmid z_1 \\ x\in A(d)}} |x|^{-n}.$$

Finally, for part (e), we provide a proof for the case when n is even. We postpone the proof for the case in which n is odd until the next section. Let π be a prime in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, and suppose that $z_1 \sim \pi^k$ for a nonnegative integer k. If $k = 0$, then z_1 is a unit and the result follows from part (a). Therefore, assume $k > 0$. Assume, for the sake of finding a contradiction, that $I_n(z_1) = I_n(z_2)$ and $|z_1| \neq |z_2|$ for some $z_2 \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$. Under this assumption, we have $|z_2|^n\delta_n(z_1) = |z_1|^n\delta_n(z_2)$. Either $N(\pi) = p$ is an integer prime or $N(\pi) = q^2$, where q is an integer prime.

First, suppose $N(\pi) = p$ is an integer prime. Then the statement $|z_2|^n\delta_n(z_1) = |z_1|^n\delta_n(z_2)$ is equivalent to $N(z_2)^{n/2}\delta_n(\pi^k) = p^{kn/2}\delta_n(z_2)$. Noting that $N(z_2)^{n/2}$, $\delta_n(\pi^k)$, and $\delta_n(z_2)$ are integers (because n is even) and that $p \nmid \delta_n(\pi^k) = 1 + p^{n/2} + \cdots + p^{kn/2}$ in $\mathbb{Z}$, we find $p^{kn/2}|N(z_2)^{n/2}$ in $\mathbb{Z}$. This implies that $p^k|N(z_2)$ in $\mathbb{Z}$, and we conclude that there exist nonnegative integers t_1, t_2 satisfying $\pi^{t_1}\overline{\pi}^{t_2}|z_2$ and $t_1 + t_2 = k$. If $\pi \sim \overline{\pi}$, then we have $\pi^k|z_2$, from which part (d) yields the desired contradiction. Otherwise, π and $\overline{\pi}$ are relatively prime, so we may use parts (b) and (d) to write

$$I_n(z_2) \geq I_n(\pi^{t_1})I_n(\overline{\pi}^{t_2}) = \frac{1 + p^{n/2} + \cdots + p^{t_1n/2}}{p^{t_1n/2}}\frac{1 + p^{n/2} + \cdots + p^{t_2n/2}}{p^{t_2n/2}}$$

$$= \frac{(1 + p^{n/2} + \cdots + p^{t_1n/2})(1 + p^{n/2} + \cdots + p^{t_2n/2})}{p^{kn/2}}$$

$$\geq \frac{1 + p^{n/2} + \cdots + p^{kn/2}}{p^{kn/2}} = I_n(\pi^k) = I_n(z_2).$$

This implies that $I_n(z_2) = I_n(\pi^{t_1}\overline{\pi}^{t_2})$, from which part (d) tells us that $z_2 \sim \pi^{t_1}\overline{\pi}^{t_2}$. Therefore, $|z_2| = |\pi^{t_1}\overline{\pi}^{t_2}| = \sqrt{p}^{t_1+t_2} = \sqrt{p}^k = |\pi^k| = |z_1|$, which we assumed was false.

Now, suppose that $N(\pi) = q^2$, where q is an integer prime (q is inert). Then the statement $|z_2|^n\delta_n(z_1) = |z_1|^n\delta_n(z_2)$ is equivalent to $N(z_2)^{n/2}\delta_n(\pi^k) = q^{kn}\delta_n(z_2)$. As before, $N(z_2)^{n/2}$, $\delta_n(\pi^k)$, and $\delta_n(z_2)$ are integers, and $q \nmid \delta_n(\pi^k) = 1 + q^n + \cdots + q^{kn}$ in $\mathbb{Z}$. Therefore, $q^{kn}|N(z_2)^{n/2}$ in $\mathbb{Z}$, so $q^{2k}|N(z_2)$ in $\mathbb{Z}$. As q is inert, this implies that $q^k|z_2$, so $z_1|z_2$ (note that $z_1 \sim \pi^k \sim q^k$). Therefore, part (d) provides the final contradiction, and the proof is complete. □

It is much easier to deal with the functions I_n when n is even than when n is odd because, when n is even, the values of $\delta_n(z)$ and $|z|^n$ are positive integers. Therefore, we will devote the next section to developing an understanding of the functions I_n for odd values of n.

3 When n is Odd

We begin by establishing some definitions and lemmas that will later prove themselves useful. Let W be the set of all square-free positive integers, and write $W = \{w_0, w_1, w_2, \ldots\}$ so that $w_0 = 1$ and $w_i < w_j$ for all nonnegative integers $i < j$. Let F be the set of all finite linear combinations of elements of W with rational coefficients. That is, $F = \{a_0 + a_1\sqrt{w_1} + \cdots + a_m\sqrt{w_m} : a_0, a_1, \ldots, a_m \in \mathbb{Q}, m \in \mathbb{N}_0\}$. For any $r \in F$, the choice of the rational coefficients is unique. More formally, if $a_0 + a_1\sqrt{w_1} + \cdots + a_m\sqrt{w_m} = b_0 + b_1\sqrt{w_1} + \cdots + b_m\sqrt{w_m}$, where $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_m \in \mathbb{Q}$, then $a_i = b_i$ for all $i \in \{0, 1, \ldots, m\}$ [2]. Note that F is a subfield of the real numbers.

Definition 3.1. For $r \in F$ and $j \in \mathbb{N}_0$, let $C_j(r)$ be the unique rational coefficient of $\sqrt{w_j}$ in the expansion of r. That is, the sequence $(C_j(r))_{j=0}^{\infty}$ is the unique infinite sequence of rational numbers that has finitely many nonzero terms and that satisfies $r = \sum_{j=0}^{\infty} C_j(r)\sqrt{w_j}$.

As an example, $C_5\left(\frac{3}{5} - \sqrt{6} + \frac{1}{3}\sqrt{7}\right) = \frac{1}{3}$ because $w_5 = 7$.

Definition 3.2. Let p be an integer prime. For $r \in F$, we say that r *has a* $\sqrt{p}$ *part* if there exists some positive integer j such that $C_j(r) \neq 0$ and $p|w_j$ (in $\mathbb{Z}$). We say that r does not have a $\sqrt{p}$ part if no such positive integer j exists.

For example, if $r = \frac{1}{2} + 3\sqrt{10}$, then r has a $\sqrt{2}$ part, and r has a $\sqrt{5}$ part. However, $\frac{1}{2} + 3\sqrt{10}$ does not have a $\sqrt{7}$ part.

Lemma 3.1. *If* $r_1, r_2 \in F$ *each do not have a* $\sqrt{p}$ *part for some integer prime* p*, then* r_1r_2 *does not have a* $\sqrt{p}$ *part.*

Proof. Suppose $p|w_j$ for some positive integer j. Then, if we let $SF(n)$ denote the square-free part of an integer n and consider the basic algebra used to multiply elements of F, we find that

$$C_j(r_1r_2) = \sum_{\substack{i_1, i_2 \in \mathbb{N}_0 \\ SF(w_{i_1}w_{i_2}) = w_j}} C_{i_1}(r_1)C_{i_2}(r_2)\sqrt{\frac{w_{i_1}w_{i_2}}{w_j}}.$$

For every pair of nonnegative integers i_1, i_2 satisfying $SF(w_{i_1}w_{i_2}) = w_j$, either $p|w_{i_1}$ or $p|w_{i_2}$. This implies that either $C_{i_1}(r_1) = 0$ or $C_{i_2}(r_2) = 0$ by the hypothesis that each of r_1 and r_2 does not have a $\sqrt{p}$ part. Thus, $C_j(r_1r_2) = 0$. As w_j was an arbitrary square-free positive integer divisible by p, we conclude that r_1r_2 does not have a $\sqrt{p}$ part. □

Lemma 3.2. *If each of* $r_1, r_2, \ldots, r_l \in F$ *does not have a* $\sqrt{p}$ *part for some integer prime* p*, then* $r_1r_2\cdots r_l$ *does not have a* $\sqrt{p}$ *part.*

Proof. The desired result follows immediately from repeated use of Lemma 3.1. □

Lemma 3.3. *If* $r_1 \in F$ *has a* $\sqrt{p}$ *part and* $r_2 \in F\backslash\{0\}$ *does not have a* $\sqrt{p}$ *part for some integer prime* p*, then* r_1r_2 *has a* $\sqrt{p}$ *part.*

Proof. Write $r_1 = r_3 + \sum_{i=1}^{k} a_i\sqrt{x_i}$, where $r_3 \in F$ does not have a $\sqrt{p}$ part and, for all distinct $i, j \in \{1, 2, \ldots, k\}$, we have $a_i \in \mathbb{Q}\backslash\{0\}$, $x_i \in W$, $p|x_i$ in $\mathbb{Z}$, and $x_i \neq x_j$. If we write $v_i = \frac{x_i}{p}$ for all $i \in \{1, 2, \ldots, k\}$, then each v_i is a square-free positive integer that is not divisible by p. Therefore,

$r_1r_2 = \left(r_3 + \sqrt{p}\sum_{i=1}^{k} a_i\sqrt{v_i}\right) r_2 = r_2r_4\sqrt{p} + r_2r_3$, where $r_4 = \sum_{i=1}^{k} a_i\sqrt{v_i}$. By the hypothesis that r_1 has a $\sqrt{p}$ part, $r_4 \neq 0$. As each of r_2, r_4 is nonzero and does not have a $\sqrt{p}$ part, Lemma 3.1 guarantees that r_2r_4 is nonzero and does not have a $\sqrt{p}$ part. Now, it is easy to see that this implies that $\sqrt{p}r_2r_4$ has a $\sqrt{p}$ part. Furthermore, each of r_2, r_3 does not have a $\sqrt{p}$ part, so Lemma 3.1 tells us that r_2r_3 does not have a $\sqrt{p}$ part. Thus, it is clear that $r_2r_4\sqrt{p} + r_2r_3$ has a $\sqrt{p}$ part, so the proof is complete. □

Lemma 3.4. *Let us fix $d \in K$ and work in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. Let π be a prime such that $N(\pi) = p$ is an integer prime. If n is an odd positive integer and $\pi|z$ for some $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$, then $I_n(z) \in F$ and $I_n(z)$ has a $\sqrt{p}$ part.*

Proof. It is clear that $I_n(z) \in F$ (this is also true for positive even integer values of n). Write $z \sim \pi^{\alpha}\overline{\pi}^{\beta}\prod_{j=1}^{r}\pi_j^{\alpha_j}$, where, for all distinct $j, k \in \{1, 2, \ldots, r\}$, π_j is prime, $N(\pi_j) \neq p$, α_j is a positive integer, and $\pi_j \not\sim \pi_k$. Fix some $j \in \{1, 2, \ldots, r\}$. If π_j is associated to an inert integer prime, then $I_n(\pi_j^{\alpha_j}) \in \mathbb{Q}$, so $I_n(\pi_j^{\alpha_j})$ does not have a $\sqrt{p}$ part. If $N(\pi_j) = p_0$ for some integer prime p_0, then $I_n(\pi_j^{\alpha_j}) = a + b\sqrt{p_0}$ for some $a, b \in \mathbb{Q}$. Again, we conclude that $I_n(\pi_j^{\alpha_j})$ does not have a $\sqrt{p}$ part because $p_0 \neq p$. Writing $x = \prod_{j=1}^{r}\pi_j^{\alpha_j}$, Lemma 3.2 and the multiplicativity of I_n guarantee that $I_n(x)$ does not have a $\sqrt{p}$ part. We now consider two cases.

First, consider the case in which p ramifies in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ (meaning $\pi \sim \overline{\pi}$). Then $z \sim \pi^{\alpha+\beta}x$. Using part (c) of Theorem 2.2, we have $I_n(\pi^{\alpha+\beta}) = \delta_{-n}(\pi^{\alpha+\beta}) =$
$\sum_{m=0}^{\alpha+\beta}\frac{1}{|\pi^m|^n} = \sum_{m=0}^{\alpha+\beta}\frac{1}{\sqrt{p}^{mn}} = t_1 + t_2\sqrt{p}$, where t_1 and t_2 are positive rational numbers. Thus, $I_n(\pi^{\alpha+\beta})$ has a $\sqrt{p}$ part, so Lemma 3.3 guarantees that $I_n(z)$ has a $\sqrt{p}$ part.

Next, consider the case in which p splits in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ (meaning $\pi \not\sim \overline{\pi}$). Then we have $I_n(\pi^{\alpha}\overline{\pi}^{\beta}) = \delta_{-n}(\pi^{\alpha})\delta_{-n}(\overline{\pi}^{\beta}) = \left(\sum_{m=0}^{\alpha}\frac{1}{|\pi^m|^n}\right)\left(\sum_{m=0}^{\beta}\frac{1}{|\overline{\pi}^m|^n}\right) =$
$\left(\sum_{m=0}^{\alpha}\frac{1}{\sqrt{p}^{mn}}\right)\left(\sum_{m=0}^{\beta}\frac{1}{\sqrt{p}^{mn}}\right) = (u_1 + u_2\sqrt{p})(u_3 + u_4\sqrt{p})$, where u_1, u_2, u_3, u_4 are positive rational numbers. Then $(u_1 + u_2\sqrt{p})(u_3 + u_4\sqrt{p}) = u_1u_3 + pu_2u_4 +$

$(u_1u_4 + u_2u_3)\sqrt{p}$. As $u_1u_4 + u_2u_3 > 0$, $I_n(\pi^\alpha \overline{\pi}^\beta)$ has a $\sqrt{p}$ part. Once again, Lemma 3.3 guarantees that $I_n(z)$ has a $\sqrt{p}$ part. □

Lemma 3.5. *Let p be an integer prime, and let $m_1, m_2, \beta_1, \beta_2$ be nonnegative integers satisfying $(p^{m_1}+p^{m_2})(p^{\beta_1+\beta_2+1}+1) = (p^{\beta_1}+p^{\beta_2})(p^{m_1+m_2+1}+1)$. Then either $m_1 = \beta_1$ and $m_2 = \beta_2$ or $m_1 = \beta_2$ and $m_2 = \beta_1$.*

Proof. Without loss of generality, we may write $m_1 = \min(m_1, m_2, \beta_1, \beta_2)$. We may also assume that $\beta_1 \leq \beta_2$ so that it suffices to show that $m_1 = \beta_1$ and $m_2 = \beta_2$. Dividing each side of the given equation by p^{m_1}, we have

$$(1 + p^{m_2-m_1})(p^{\beta_1+\beta_2+1} + 1) = (p^{\beta_1-m_1} + p^{\beta_2-m_1})(p^{m_1+m_2+1} + 1). \tag{3.1}$$

Suppose $m_1 = \beta_1$. Then (3.1) becomes $(1 + p^{m_2-m_1})(p^{m_1+\beta_2+1} + 1) = (1 + p^{\beta_2-m_1})(p^{m_1+m_2+1} + 1)$. Now, define a function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \dfrac{p^{m_1+x+1} + 1}{1 + p^{x-m_1}}$. We may differentiate to get

$$f'(x) = \frac{(p^{m_1+x})(p^{2m_1+1} - 1)}{(p^x + p^{m_1})^2} \log p > 0,$$

so f is one-to-one. As $f(m_2) = f(\beta_2)$, we have $m_2 = \beta_2$. Therefore, we only need to show that $m_1 = \beta_1$.

Suppose $p \neq 2$. Then, if $m_1 < \beta_1$, we may read (3.1) modulo p to reach a contradiction. Thus, if $p \neq 2$, we are done. Now, suppose $p = 2$ and $m_1 < \beta_1$ so that (3.1) becomes

$$(1 + 2^{m_2-m_1})(2^{\beta_1+\beta_2+1} + 1) = (2^{\beta_1-m_1} + 2^{\beta_2-m_1})(2^{m_1+m_2+1} + 1). \tag{3.2}$$

The right-hand side of (3.2) is even, which implies that we must have $m_1 = m_2$ so that $1+2^{m_2-m_1} = 2$. Dividing each side of (3.2) by 2 yields $2^{\beta_1+\beta_2+1} + 1 = (2^{\beta_1-m_1-1} + 2^{\beta_2-m_1-1})(2^{2m_1+1} + 1)$. As the left-hand side of this last equation is odd, we must have $\beta_1 = m_1 + 1$. Therefore, $2^{\beta_1+\beta_2+1} + 1 = (1 + 2^{\beta_2-\beta_1})(2^{2\beta_1-1}+1) = 2^{\beta_1+\beta_2-1}+2^{\beta_2-\beta_1}+2^{2\beta_1-1}+1$. If we subtract $2^{\beta_1+\beta_2-1}+1$ from each side of this last equation, we get $3 \cdot 2^{\beta_1+\beta_2-1} = 2^{\beta_2-\beta_1} + 2^{2\beta_1-1}$. However, $3 \cdot 2^{\beta_1+\beta_2-1} > 2^{\beta_1+\beta_2-1} + 2^{\beta_1+\beta_2-1} > 2^{\beta_2-\beta_1} + 2^{2\beta_1-1}$, so we have reached our final contradiction. This completes the proof. □

We now possess the tools necessary to complete the proof of part (e) of Theorem 2.2. We do so in the following two theorems.

Theorem 3.1. *Let us work in a ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $d \in K$, and let n be an odd positive integer. Let π be a prime such that $\pi \sim \overline{\pi}$, and let k be a positive integer. Then π^k is n-powerfully solitary in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.*

Proof. We suppose, for the sake of finding a contradiction, that there exists $x \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \setminus \{0\}$ such that $|x| \neq |\pi^k|$ and $I_n(x) = I_n(\pi^k)$. Suppose that π_0 is a prime such that $\pi_0 | x$ and $N(\pi_0) = p_0$ is an integer prime. Then, by Lemma 3.4, $I_n(x)$ has a $\sqrt{p_0}$ part. This implies that $I_n(\pi^k)$ has a $\sqrt{p_0}$ part. However, if $N(\pi) = p$, where p is an integer prime, then $I_n(\pi^k) = \sum_{m=0}^{k} \frac{1}{\sqrt{p}^{mn}} = t_1 + t_2\sqrt{p}$ for some $t_1, t_2 \in \mathbb{Q}$. Hence, we find that $p_0 = p$, which means that $\pi_0 \sim \pi$. On the other hand, if π is associated to an inert integer prime q, then $I_n(\pi^k) \in \mathbb{Q}$. Therefore, if a prime that is not associated to π divides x, that prime must be associated to an inert integer prime. We now consider two cases.

Case 1: In this case, $\pi \sim q$, where q is an inert integer prime. This implies that all primes dividing x must be associated to inert integer primes, so $\delta_n(x)$ and $|x|$ are integers. From $I_n(x) = I_n(\pi^k)$ and $|\pi^k|^n = q^{kn}$, we have $\delta_n(x) q^{kn} = \delta_n(\pi^k)|x|^n$. We know that $\delta_n(\pi^k) = \sum_{j=0}^{k} |\pi^j|^n = 1 + \sum_{j=1}^{k} q^{jn}$, so $q \nmid \delta_n(\pi^k)$ in $\mathbb{Z}$. Therefore, q^{kn} divides $|x|^n$ in $\mathbb{Z}$, so q^k divides $|x|$ in $\mathbb{Z}$. We conclude that $q^k | x$, so $\pi^k | x$. However, part (*d*) of Theorem 2.2 tells us that this is a contradiction.

Case 2: In this case, $N(\pi) = p$ is an integer prime. Because all of the prime divisors of x that are not associated to π must be associated to inert integer primes, we may write $x \sim \pi^\alpha \prod_{j=1}^{t} q_j^{\beta_j}$, where $\alpha \in \mathbb{N}_0$ and, for each $j \in \{1, 2, \ldots, t\}$, q_j is an inert integer prime and β_j is a positive integer. Note that $\alpha \geq 1$ because $I_n(\pi^k)$ has a $\sqrt{p}$ part, which implies that $I_n(x)$ has a $\sqrt{p}$ part. Also, $\alpha < k$ because, otherwise, $\pi^k | x$, from which part (*d*) of Theorem 2.2 yields a contradiction. We have

$$I_n(\pi^k) = \frac{\sum_{l=0}^{k} \sqrt{p}^{ln}}{\sqrt{p}^{kn}} = \frac{\sqrt{p}^{(k+1)n} - 1}{\sqrt{p}^{kn}(\sqrt{p}^n - 1)},$$

and

$$I_n(\pi^\alpha) = \frac{\sum_{l=0}^{\alpha} \sqrt{p}^{ln}}{\sqrt{p}^{\alpha n}} = \frac{\sqrt{p}^{(\alpha+1)n} - 1}{\sqrt{p}^{\alpha n}(\sqrt{p}^n - 1)}.$$

Now, $\frac{I_n(\pi^k)}{I_n(\pi^\alpha)} = I_n\left(\prod_{j=1}^{t} q_j^{\beta_j}\right) \in \mathbb{Q}$ because each integer prime q_j is inert. This implies that $(p^{(\alpha+1)n}-1)\dfrac{I_n(\pi^k)}{I_n(\pi^\alpha)} \in \mathbb{Q}$. We have

$$(p^{(\alpha+1)n}-1)\frac{I_n(\pi^k)}{I_n(\pi^\alpha)} = (p^{(\alpha+1)n}-1)\frac{\sqrt{p}^{(k+1)n-1}}{(\sqrt{p}^{(\alpha+1)n}-1)\sqrt{p}^{(k-\alpha)n}}$$

$$= (\sqrt{p}^{(\alpha+1)n}-1)(\sqrt{p}^{(\alpha+1)n}+1)\frac{\sqrt{p}^{(k+1)n}-1}{(\sqrt{p}^{(\alpha+1)n}-1)\sqrt{p}^{(k-\alpha)n}}$$

$$= \frac{(\sqrt{p}^{(k+1)n}-1)(\sqrt{p}^{(\alpha+1)n}+1)}{\sqrt{p}^{(k-\alpha)n}} \in \mathbb{Q}.$$

If k is odd, then $\sqrt{p}^{(k+1)n}-1$ is rational, which implies that α must also be odd. Similarly, if α is odd, then k must be odd. Therefore, k and α have the same parities, which implies that $\sqrt{p}^{(k-\alpha)n}$ is rational. This implies $(\sqrt{p}^{(k+1)n}-1)(\sqrt{p}^{(\alpha+1)n}+1) \in \mathbb{Q}$. We clearly have a contradiction if k and α are both even, so they must both be odd. As k is odd, we have

$$I_n(\pi^k) = \delta_{-n}(\pi^k) = \sum_{l=0}^{k}\frac{1}{\sqrt{p}^{ln}} = \left(\sum_{m=0}^{\frac{k-1}{2}}\frac{1}{\sqrt{p}^{2mn}}\right) + \left(\sum_{m=0}^{\frac{k-1}{2}}\frac{1}{\sqrt{p}^{2mn}}\right)\left(\frac{1}{\sqrt{p}^{n}}\right)$$

$$= \left(\sum_{m=0}^{\frac{k-1}{2}}\frac{1}{p^{mn}}\right)\left(1+\frac{1}{\sqrt{p}^{n}}\right) = \frac{h_1}{p^n-1}\left(1+\frac{1}{\sqrt{p}^{n}}\right),$$

where

$$h_1 = \left(\sum_{m=0}^{\frac{k-1}{2}}\frac{1}{p^{mn}}\right)(p^n-1) = \frac{p^{\frac{k+1}{2}n}-1}{p^{\frac{k-1}{2}n}}.$$

Similarly, if we write $h_2 = \dfrac{p^{\frac{\alpha+1}{2}n}-1}{p^{\frac{\alpha-1}{2}n}}$, then we have $I_n(\pi^\alpha) = \dfrac{h_2}{p^n-1}\left(1+\dfrac{1}{\sqrt{p}^{n}}\right)$. Now,

$$I_n\left(\prod_{j=1}^{t} q_j^{\beta_j}\right) = \frac{I_n(\pi^k)}{I_n(\pi^\alpha)} = \frac{h_1}{h_2} = \frac{p^{\frac{k+1}{2}n}-1}{p^{\frac{k-\alpha}{2}n}\left(p^{\frac{\alpha+1}{2}n}-1\right)},$$

so

$$\left[\delta_n\left(\prod_{j=1}^{t} q_j^{\beta_j}\right)\right]\left[p^{\frac{k-\alpha}{2}n}\right]\left[p^{\frac{\alpha+1}{2}n}-1\right] = \left[\left|\prod_{j=1}^{t} q_j^{\beta_j}\right|^n\right]\left[p^{\frac{k+1}{2}n}-1\right].$$

Notice that each bracketed expression in this last equation is an integer, and notice that p divides the left-hand side in $\mathbb{Z}$. However, p does not divide the right-hand side in $\mathbb{Z}$, so we have a contradiction. $\square$

We now only have to prove part (e) of Theorem 2.2 for the case in which n is odd and $\pi \not\sim \overline{\pi}$. We do so as a corollary of the following more general theorem.

Theorem 3.2. *Let us work in a ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $d \in K$, and let n be an odd positive integer. Let π be a prime such that $\pi \not\sim \overline{\pi}$, and let k_1, k_2 be nonnegative integers. Then $\pi^{k_1}\overline{\pi}^{k_2}$ is n-powerfully solitary in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ unless, possibly, if k_1 and k_2 are both odd. In the case that k_1 and k_2 are both odd, any friend of $\pi^{k_1}\overline{\pi}^{k_2}$, say x, must satisfy $x \sim \pi^{\alpha_1}\overline{\pi}^{\alpha_2}\prod_{j=1}^{t} q_j^{\gamma_j}$, where α_1, α_2 are odd positive integers and, for each $j \in \{1, 2, \dots, t\}$, q_j is an inert integer prime and γ_j is a positive integer.*

Proof. First note that Fact 1.2 tells us that $N(\pi) = N(\overline{\pi}) = p$, where p is an integer prime.

We suppose, for the sake of finding a contradiction, that there exists $x \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$ such that $|x| \neq |\pi^{k_1}\overline{\pi}^{k_2}|$ and $I_n(x) = I_n(\pi^{k_1}\overline{\pi}^{k_2})$. Suppose that π_0 is a prime such that $\pi_0|x$ and $N(\pi_0) = p_0$ is an integer prime. Then, by Lemma 3.4, $I_n(x)$ has a $\sqrt{p_0}$ part. This implies that $I_n(\pi^{k_1}\overline{\pi}^{k_2})$ has a $\sqrt{p_0}$ part. However, as $N(\pi) = N(\overline{\pi}) = p$, we must have $I_n(\pi^{k_1}\overline{\pi}^{k_2}) = I_n(\pi^{k_1})I_n(\overline{\pi}^{k_2}) = \left(\sum_{m=0}^{k_1} \frac{1}{\sqrt{p}^{mn}}\right)\left(\sum_{m=0}^{k_2} \frac{1}{\sqrt{p}^{mn}}\right) = t_1 + t_2\sqrt{p}$ for some $t_1, t_2 \in \mathbb{Q}$, so we find that $p_0 = p$. Therefore, if a prime that is not associated to π or $\overline{\pi}$ divides x, that prime must be associated to an inert integer prime. Hence, we may write $x \sim \pi^{\alpha_1}\overline{\pi}^{\alpha_2}\prod_{j=1}^{t} q_j^{\gamma_j}$, where $\alpha_1, \alpha_2 \in \mathbb{N}_0$ and, for each $j \in \{1, 2, \dots, t\}$, q_j is an inert integer prime and γ_j is a positive integer.

We have

$$I_n(\pi^{k_1}) = \frac{\sum_{l=0}^{k_1} \sqrt{p}^{ln}}{\sqrt{p}^{k_1 n}} = \frac{\sqrt{p}^{(k_1+1)n} - 1}{\sqrt{p}^{k_1 n}(\sqrt{p}^n - 1)},$$

$$I_n(\overline{\pi}^{k_2}) = \frac{\sum_{l=0}^{k_2} \sqrt{p}^{ln}}{\sqrt{p}^{k_2 n}} = \frac{\sqrt{p}^{(k_2+1)n} - 1}{\sqrt{p}^{k_2 n}(\sqrt{p}^n - 1)},$$

$$I_n(\pi^{\alpha_1}) = \frac{\sum_{l=0}^{\alpha_1} \sqrt{p}^{ln}}{\sqrt{p}^{\alpha_1 n}} = \frac{\sqrt{p}^{(\alpha_1+1)n} - 1}{\sqrt{p}^{\alpha_1 n}(\sqrt{p}^n - 1)},$$

and

$$I_n(\overline{\pi}^{\alpha_2}) = \frac{\sum_{l=0}^{\alpha_2} \sqrt{p}^{ln}}{\sqrt{p}^{\alpha_2 n}} = \frac{\sqrt{p}^{(\alpha_2+1)n} - 1}{\sqrt{p}^{\alpha_2 n}(\sqrt{p}^n - 1)}.$$

Now, $\dfrac{I_n(\pi^{k_1})I_n(\overline{\pi}^{k_2})}{I_n(\pi^{\alpha_1})I_n(\overline{\pi}^{\alpha_2})} = \dfrac{I_n(\pi^{k_1}\overline{\pi}^{k_2})}{I_n(\pi^{\alpha_1}\overline{\pi}^{\alpha_2})} = I_n\left(\prod_{j=1}^{t} q_j^{\gamma_j}\right) \in \mathbb{Q}$ because each integer prime q_j is inert. This implies that $(p^{(\alpha_1+1)n} - 1)(p^{(\alpha_2+1)n} - 1)\dfrac{I_n(\pi^{k_1})I_n(\overline{\pi}^{k_2})}{I_n(\pi^{\alpha_1})I_n(\overline{\pi}^{\alpha_2})} \in \mathbb{Q}$. We have

$$(p^{(\alpha_1+1)n} - 1)(p^{(\alpha_2+1)n} - 1)\frac{I_n(\pi^{k_1})I_n(\overline{\pi}^{k_2})}{I_n(\pi^{\alpha_1})I_n(\overline{\pi}^{\alpha_2})}$$

$$= (p^{(\alpha_1+1)n} - 1)(p^{(\alpha_2+1)n} - 1)\frac{(\sqrt{p}^{(k_1+1)n} - 1)(\sqrt{p}^{(k_2+1)n} - 1)}{(\sqrt{p}^{(\alpha_1+1)n} - 1)(\sqrt{p}^{(\alpha_2+1)n} - 1)\sqrt{p}^{(k_1+k_2-\alpha_1-\alpha_2)n}}$$

$$= \frac{(\sqrt{p}^{(k_1+1)n} - 1)(\sqrt{p}^{(k_2+1)n} - 1)(\sqrt{p}^{(\alpha_1+1)n} + 1)(\sqrt{p}^{(\alpha_2+1)n} + 1)}{\sqrt{p}^{(k_1+k_2-\alpha_1-\alpha_2)n}} \in \mathbb{Q}.$$

We now consider several cases. In what follows, we will write $m_1 = \dfrac{(k_1+1)n - 1}{2}$, $m_2 = \dfrac{(k_2+1)n - 1}{2}$, $\beta_1 = \dfrac{(\alpha_1+1)n - 1}{2}$, and $\beta_2 = \dfrac{(\alpha_2+1)n - 1}{2}$. This will simplify notation because, for example, if k_1 is even, then $\sqrt{p}^{(k_1+1)n} = p^{m_1}\sqrt{p}$ and m_1 is a nonnegative integer.

Case 1: $\alpha_1 \not\equiv \alpha_2 \equiv k_1 \equiv k_2 \equiv 1 \mod 2$. In this case, $(\sqrt{p}^{(k_1+1)n} - 1)(\sqrt{p}^{(k_2+1)n} - 1)(\sqrt{p}^{(\alpha_2+1)n} + 1) \in \mathbb{Q}$, so $\dfrac{\sqrt{p}^{(\alpha_1+1)n} + 1}{\sqrt{p}^{(k_1+k_2-\alpha_1-\alpha_2)n}} \in \mathbb{Q}$. However, this is impossible because $(\alpha_1+1)n$ is odd. By the same argument, we may show that it is impossible to have exactly one of $k_1, k_2, \alpha_1, \alpha_2$ be even.

Case 2: $\alpha_1 \not\equiv \alpha_2 \equiv k_1 \equiv k_2 \equiv 0 \mod 2$. In this case, $\sqrt{p}^{(\alpha_1+1)n} - 1 \in \mathbb{Q}$, and $\sqrt{p}^{(k_1+k_2-\alpha_1-\alpha_2)n} = \mu\sqrt{p}$ for some $\mu \in \mathbb{Q}$. This implies that

$$(\sqrt{p}^{(k_1+1)n} - 1)(\sqrt{p}^{(k_2+1)n} - 1)(\sqrt{p}^{(\alpha_2+1)n} + 1)$$
$$= (p^{m_1}\sqrt{p} - 1)(p^{m_2}\sqrt{p} - 1)(p^{\beta_2}\sqrt{p} + 1) = \lambda\sqrt{p}$$

for some $\lambda \in \mathbb{Q}$. We may expand to get

$$(p^{m_1}\sqrt{p} - 1)(p^{m_2}\sqrt{p} - 1)(p^{\beta_2}\sqrt{p} + 1)$$

$$= ((p^{m_1+m_2+1}+1) - (p^{m_1}+p^{m_2})\sqrt{p})(p^{\beta_2}\sqrt{p}+1)$$
$$= (p^{m_1+m_2+1}+1-p^{\beta_2+1}(p^{m_1}+p^{m_2})) + (p^{\beta_2}(p^{m_1+m_2+1}+1) - (p^{m_1}+p^{m_2}))\sqrt{p}.$$

As $m_1, m_2, \beta_2 \in \mathbb{N}_0$, we find that $p^{m_1+m_2+1}+1-p^{\beta_2+1}(p^{m_1}+p^{m_2})$ and $p^{\beta_2}(p^{m_1+m_2+1}+1)-(p^{m_1}+p^{m_2})$ are integers. Therefore, from the equation $(p^{m_1+m_2+1}+1-p^{\beta_2+1}(p^{m_1}+p^{m_2}))+(p^{\beta_2}(p^{m_1+m_2+1}+1)-(p^{m_1}+p^{m_2}))\sqrt{p} = \lambda\sqrt{p}$, we have $p^{m_1+m_2+1}+1-p^{\beta_2+1}(p^{m_1}+p^{m_2}) = 0$. Reading this last equation modulo p, we have a contradiction. The same argument eliminates the case $\alpha_2 \not\equiv \alpha_1 \equiv k_1 \equiv k_2 \equiv 0 \mod 2$.

Case 3: $k_1 \not\equiv k_2 \equiv \alpha_1 \equiv \alpha_2 \equiv 0 \mod 2$. In this case, $\sqrt{p}^{(k_1+1)n} - 1 \in \mathbb{Q}$, and $\sqrt{p}^{(k_1+k_2-\alpha_1-\alpha_2)n} = \mu\sqrt{p}$ for some $\mu \in \mathbb{Q}$. This implies that

$$(\sqrt{p}^{(k_2+1)n}-1)(\sqrt{p}^{(\alpha_1+1)n}+1)(\sqrt{p}^{(\alpha_2+1)n}+1)$$
$$= (p^{m_2}\sqrt{p}-1)(p^{\beta_1}\sqrt{p}+1)(p^{\beta_2}\sqrt{p}+1) = \lambda\sqrt{p}$$

for some $\lambda \in \mathbb{Q}$. We may expand just as we did in Case 2, and we will find $p^{m_1+m_2+1}+1+p^{\beta_2+1}(p^{m_1}+p^{m_2}) = 0$, which is clearly a contradiction. This same argument eliminates the case $k_2 \not\equiv k_1 \equiv \alpha_1 \equiv \alpha_2 \equiv 0 \mod 2$.

Case 4: $k_1 \equiv k_2 \equiv 1 \mod 2$, and $\alpha_1 \equiv \alpha_2 \equiv 0 \mod 2$. In this case, $\dfrac{(\sqrt{p}^{(k_1+1)n}-1)(\sqrt{p}^{(k_2+1)n}-1)}{\sqrt{p}^{(k_1+k_2-\alpha_1-\alpha_2)n}} \in \mathbb{Q}$, so we must have $(\sqrt{p}^{(\alpha_1+1)n}+1)(\sqrt{p}^{(\alpha_2+1)n}+1) = (p^{\beta_1}\sqrt{p}+1)(p^{\beta_2}\sqrt{p}+1) \in \mathbb{Q}$. However, this is impossible because β_1 and β_2 are nonnegative integers.

Case 5: $k_1 \equiv k_2 \equiv 0 \mod 2$, and $\alpha_1 \equiv \alpha_2 \equiv 1 \mod 2$. In this case, $\dfrac{(\sqrt{p}^{(\alpha_1+1)n}+1)(\sqrt{p}^{(\alpha_2+1)n}+1)}{\sqrt{p}^{(k_1+k_2-\alpha_1-\alpha_2)n}} \in \mathbb{Q}$, so we must have $(\sqrt{p}^{(k_1+1)n}-1)(\sqrt{p}^{(k_2+1)n}-1) = (p^{m_1}\sqrt{p}-1)(p^{m_2}\sqrt{p}-1) \in \mathbb{Q}$. However, this is impossible because m_1 and m_2 are nonnegative integers.

Case 6: $k_1 \equiv k_2 \equiv \alpha_1 \equiv \alpha_2 \equiv 0 \mod 2$. In this case, $\sqrt{p}^{(k_1+k_2-\alpha_1-\alpha_2)n} \in \mathbb{Q}$, so

$$(\sqrt{p}^{(k_1+1)n}-1)(\sqrt{p}^{(k_2+1)n}-1)(\sqrt{p}^{(\alpha_1+1)n}+1)(\sqrt{p}^{(\alpha_2+1)n}+1)$$
$$= (p^{m_1}\sqrt{p}-1)(p^{m_2}\sqrt{p}-1)(p^{\beta_1}\sqrt{p}+1)(p^{\beta_2}\sqrt{p}+1) \in \mathbb{Q}.$$

One may verify that, after expanding this last expression and noting that m_1, m_2, β_1, and β_2 must be positive integers, we arrive at the requirement

$(p^{m_1}+p^{m_2})(p^{\beta_1+\beta_2+1}+1) = (p^{\beta_1}+p^{\beta_2})(p^{m_1+m_2+1}+1)$. Lemma 3.5 then guarantees that either $m_1=\beta_1$ and $m_2=\beta_2$ or $m_1=\beta_2$ and $m_2=\beta_1$, which means that either $k_1=\alpha_1$ and $k_2=\alpha_2$ or $k_1=\alpha_2$ and $k_2=\alpha_1$. Then $\dfrac{I_n(\pi^{k_1})I_n(\overline{\pi}^{k_2})}{I_n(\pi^{\alpha_1})I_n(\overline{\pi}^{\alpha_2})} = I_n\left(\prod_{j=1}^{t} q_j^{\gamma_j}\right) = 1$, which implies that $\prod_{j=1}^{t} q_j^{\gamma_j}$ is a unit. However, we then find that $|\pi^{k_1}\overline{\pi}^{k_2}| = |\pi^{\alpha_1}\overline{\pi}^{\alpha_2}| = |x|$, which we originally assumed was not true. Therefore, this case yields a contradiction.

Case 7: $k_1 \equiv \alpha_1 \equiv 1 \mod 2$ and $k_2 \equiv \alpha_2 \equiv 0 \mod 2$. In this case, $\dfrac{(\sqrt{p}^{(k_1+1)n}-1)(\sqrt{p}^{(\alpha_1+1)n}+1)}{\sqrt{p}^{(k_1+k_2-\alpha_1-\alpha_2)n}} \in \mathbb{Q}$, so we must have $(\sqrt{p}^{(k_2+1)n}-1)(\sqrt{p}^{(\alpha_2+1)n}+1) = (p^{m_2}\sqrt{p}-1)(p^{\beta_2}\sqrt{p}+1) \in \mathbb{Q}$. Writing $(p^{m_2}\sqrt{p}-1)(p^{\beta_2}\sqrt{p}+1) = (p^{m_2+\beta_2+1}-1)+(p^{m_2}-p^{\beta_2})\sqrt{p}$ and noting that m_2 and β_2 are nonnegative integers, we find that $m_2=\beta_2$. Therefore, $k_2=\alpha_2$, so $I_n\left(\prod_{j=1}^{t} q_j^{\gamma_j}\right) = \dfrac{I_n(\pi^{k_1})I_n(\overline{\pi}^{k_2})}{I_n(\pi^{\alpha_1})I_n(\overline{\pi}^{\alpha_2})} = \dfrac{I_n(\pi^{k_1})}{I_n(\pi^{\alpha_1})}$. Because $I_n\left(\prod_{j=1}^{t} q_j^{\gamma_j}\right) > 1$, we see that $\alpha_1 < k_1$. As k_1 is odd, we have

$$I_n(\pi^{k_1}) = \delta_{-n}(\pi^{k_1}) = \sum_{l=0}^{k_1}\frac{1}{\sqrt{p}^{ln}} = \left(\sum_{r=0}^{\frac{k_1-1}{2}}\frac{1}{\sqrt{p}^{2rn}}\right) + \left(\sum_{r=0}^{\frac{k_1-1}{2}}\frac{1}{\sqrt{p}^{2rn}}\right)\left(\frac{1}{\sqrt{p}^{n}}\right)$$

$$= \left(\sum_{r=0}^{\frac{k_1-1}{2}}\frac{1}{p^{rn}}\right)\left(1+\frac{1}{\sqrt{p}^{n}}\right) = \frac{h_1}{p^n-1}\left(1+\frac{1}{\sqrt{p}^{n}}\right),$$

where

$$h_1 = \left(\sum_{r=0}^{\frac{k_1-1}{2}}\frac{1}{p^{rn}}\right)(p^n-1) = \frac{p^{\frac{k_1+1}{2}n}-1}{p^{\frac{k_1-1}{2}n}}.$$

Similarly, if we write $h_2 = \dfrac{p^{\frac{\alpha_1+1}{2}n}-1}{p^{\frac{\alpha_1-1}{2}n}}$, then we have $I_n(\pi^{\alpha_1}) = \dfrac{h_2}{p^n-1}(1+\dfrac{1}{\sqrt{p}^{n}})$. Now,

$$I_n\left(\prod_{j=1}^{t} q_j^{\gamma_j}\right) = \frac{I_n(\pi^{k_1})}{I_n(\pi^{\alpha_1})} = \frac{h_1}{h_2} = \frac{p^{\frac{k_1+1}{2}n}-1}{p^{\frac{k_1-\alpha_1}{2}n}(p^{\frac{\alpha_1+1}{2}n}-1)},$$

so

$$\left[\delta_n\left(\prod_{j=1}^{t} q_j^{\gamma_j}\right)\right]\left[p^{\frac{k_1-\alpha_1}{2}n}\right]\left[p^{\frac{\alpha_1+1}{2}n}-1\right]=\left[\left|\prod_{j=1}^{t} q_j^{\gamma_j}\right|^n\right]\left[p^{\frac{k_1+1}{2}n}-1\right].$$

Now, each bracketed part of this last equation is an integer, and p divides the left-hand side in $\mathbb{Z}$. However, p does not divide the right-hand side in $\mathbb{Z}$, so we have a contradiction. We may use this same argument to find contradictions in the three other cases in which $k_1 \not\equiv k_2 \mod 2$ and $\alpha_1 \not\equiv \alpha_2 \mod 2$.

One may check that we have found contradictions for all of the possible choices of parities of k_1, k_2, α_1, and α_2 except the case in which all four are odd. Therefore, the proof is complete. □

Corollary 3.1. *Let $d \in K$, and let $k, n \in \mathbb{N}$ with n odd. If π is a prime in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ such that $\pi \not\sim \overline{\pi}$, then π^k is n-powerfully solitary in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.*

Proof. Setting $k_1 = k$ and $k_2 = 0$ in Theorem 3.1, we find that π^k is n-powerfully solitary in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ because k_2 is even. □

Corollary 3.2. *Let $d \in K$, and let p be an integer prime. Let k be a positive integer that is either even or equal to 1, and let n be an odd positive integer. If $z \sim p^k$, then z is n-powerfully solitary in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.*

Proof. If p is inert or ramified in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, then $z \sim \pi^\alpha$ for some prime π and some positive integer α. Therefore, z is n-powerfully solitary in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ by part (e) of Theorem 2.2. If p splits in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ and $k = 1$, then $z \sim \pi\overline{\pi}$. Therefore, by Theorem 3.2, any friend of z, say x, must satisfy $x \sim \pi^{\alpha_1}\overline{\pi}^{\alpha_2}\prod_{j=1}^{t} q_j^{\gamma_j}$, where α_1, α_2 are odd positive integers and, for each $j \in \{1, 2, \dots, t\}$, q_j is an inert integer prime and γ_j is a positive integer. However, this implies that $\pi\overline{\pi}|x$, so $z|x$. This is a contradiction. Finally, if p splits in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ and k is even, then $z \sim \pi^k\overline{\pi}^k$. As k is even, the result follows from Theorem 3.2. □

Note that Corollary 3.1 delivers the final blow in the proof of part (e) of Theorem 2.2.

4 Concluding Remarks and Open Questions

After the introduction of our generalization of the abundancy index, we quickly become inundated with new questions. We pose a few such problems, acknowledging that their difficulties could easily span a large gamut.

To begin, we note that we have focused exclusively on rings $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $d \in K$. One could generalize the definitions presented here to the other quadratic integer rings. While complications could surely arise in rings without unique factorization, generalizing the abundancy index to unique factorization domains $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $d > 0$ does not seem to be a highly formidable task.

Even if we continue to restrict our attention to the rings $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $d \in K$, we may ask some interesting questions. For example, for a given n, what are some examples of n-powerful friends in these rings? We might also ask which numbers (or which types of numbers), are n-powerfully solitary for a given n. For example, the number 21 is solitary in $\mathbb{Z}$, so it is not difficult to show that 21 is also solitary in $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$. Furthermore, for a given element of some ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, we might ask to find the values of n for which this element is n-powerfully solitary.

Conjecture 4.1. *Let $d \in K$. If p is an integer prime and k is a positive integer, then p^k is n-powerfully solitary in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ for all positive integers n. As a stronger form of this conjecture, we wonder if $\pi^{\alpha_1}\overline{\pi}^{\alpha_2}$ is necessarily n-powerfully solitary in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ whenever π is a prime in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ and $\alpha_1, \alpha_2, n \in \mathbb{N}$. Note that part (e) of Theorem 2.2 and Theorem 3.2 resolve this issue for many cases.*

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