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# Some conjectures in elementary number theory

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**Abstract**

We announce a number of conjectures associated with and arising from a study of primes and irrationals in $\mathbb{R}$. All are supported by numerical verification to the extent possible.

## The Conjectures

### Bhargava factorials

For definitions and basic results dealing with Bhargava's factorial functions we refer the reader to [3], [4], [5] and [9]. Briefly, let $X \subseteq \mathbf{Z}$ be a finite or infinite set of integers. Following [5], one can define the notion of a $p$-ordering on $X$ and use it to define a set of generalized factorials of the set $X$ inductively. By definition $0!_X = 1$. Whenever $p$ a prime, we fix an element $a_0 \in X$ and, for $k \geq 1$, we select $a_k$ such that the highest power of $p$ dividing $\prod_{i=0}^{k-1} (a_k - a_i)$ is minimized. The resulting sequence of $a_i$ is then called a $p$-ordering of $X$. As one can gather from the definition, $p$-orderings are not unique, as one can vary $a_0$. On the other hand, associated with such a

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$p$-ordering of $X$ we define an associated $p$-sequence $\{\nu_k(X,p)\}_{k=1}^{\infty}$ by

$$\nu_k(X,p) = w_p(\prod_{i=0}^{k-1}(a_k - a_i)),$$

where $w_p(a)$ is, by definition, the highest power of $p$ dividing $a$ (e.g., $w_2(80) = 16$). One can show that although the $p$-ordering is not unique the associated $p$-sequence is independent of the $p$-ordering used. Since this quantity is an invariant it can be used to define generalized factorials of $X$ by setting

$$k!_X = \prod_p \nu_k(X,p), \tag{1}$$

where the (necessarily finite) product extends over all primes $p$.

**Definition 0.1.** *[13]. An abstract (or generalized) factorial is a function $!_a : \mathbb{N} \to \mathbb{Z}^+$ that satisfies the following conditions:*

1. $0!_a = 1$,
2. *For every non-negative integers $n, k$, $0 \le k \le n$ the generalized binomial coefficients*
$$\binom{n}{k}_a := \frac{n!_a}{k!_a(n-k)!_a} \in \mathbb{Z}^+,$$
3. *For every positive integer $n$, $n!$ divides $n!_a$.*

It is easy to see that the collection of all abstract factorials forms a commutative semigroup under ordinary pointwise multiplication. In fact, it is easy to see that Bhargava's factorial function is an abstract factorial. (Indeed, Hypothesis 1 of Definition 0.1 is clear by definition of the factorial in question. Hypothesis 2 of Definition 0.1 follows by the results in [5]. )

The context of these first three conjectures is the construction in [5] as applied to the ring of integers. In this case, the factorial function for the set of rational primes

$$\mathbb{P} = \{2, 3, 5, 7, 11, \dots\}$$

is given by [5]

$$n!_{\mathbb{P}} = \prod_p p^{\sum_{m=0}^{\infty}[\frac{n-1}{p^m(p-1)}]}. \tag{2}$$

We call this simply the B-factorial for the set under consideration. In the sequel, the statement "For every $n \ge 1$" means "for every integer $n \ge 1$ for which the factorials are defined".

Let $\mathbb{P}_2 \subset \mathbb{P}$ denote the subset of all twin primes, i.e., those primes of the form $p, p+2$ as usual. Let $n!_{\mathbb{P}_2}$ denote the B-factorial of the set $\mathbb{P}_2$. In the following conjectures the notation $w_p(n)$ is used to identify the highest power of $p$ that divides $n$. So, for example, if $n$ has the representation $n = 2^{a_1}\alpha$ and $(\alpha, 2) = 1$, then $w_2(n) = 2^{a_1}$.

**Conjecture 1.** *For every* $n \geq 1$*,*

$$\frac{n!_{\mathbb{P}_2}}{n!_{\mathbb{P}}} = 2\, w_2(n).$$

**NOTE.** Conjecture 1 was disproved for $n = 22$ by Vladislav Volkov [16]. Is it true that

$$\frac{n!_{\mathbb{P}_2}}{n!_{\mathbb{P}}} = 2\, w_2(n)(n+1),$$

holds for infinitely many $n$ whenever $n-1$ and $n+1$ is not a prime pair but $n+1$ is a prime?

In analogy with the preceding we let $\mathbb{P}_3 \subset \mathbb{P}$ denote that subset of all prime triplets of the form $p, p+2, p+6$. Let $n!_{\mathbb{P}_3}$ denote the B-factorial of the set $\mathbb{P}_3$.

**Conjecture 2.** *For every* $n \geq 1$*,*

$$\frac{n!_{\mathbb{P}_3}}{n!_{\mathbb{P}}} = \begin{cases} 3!\, w_2(n) w_3(n), & \text{if } n \text{ is even,} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

Next, let $\mathbb{P}_4 \subset \mathbb{P}$ denote that subset of all prime quadruplets written in the form $p, p+2, p+6, p+8$. Since $p, p+2$ and $p+6, p+8$ are both twin primes we can view $\mathbb{P}_4 \subset \mathbb{P}_2$, and so we must have $n!_{\mathbb{P}_2} | n!_{\mathbb{P}_4}$, by [[5], Lemma 13]. In fact, we claim that,

**Conjecture 3.** *For every* $n \geq 1$*,*

$$\frac{n!_{\mathbb{P}_4}}{n!_{\mathbb{P}_2}} = \begin{cases} 3\, w_3(n), & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

**NOTE.** Conjectures 2 and 3 are still open though I am not hopeful at this time on account of the counterexample to the previous conjecture.

These three conjectures have been verified using Crabbe's algorithm [11] to the limits available by the hardware. For motivation see [13].

## Prime number inequalities

Now let $p_n$ denote the n-th prime. Then, see [13],

**Conjecture 4.**

$$p_n \geq p_k + p_{n-k-1}, \qquad 1 \leq k \leq n-1, \tag{3}$$

*and all* $n \geq 2$.

**NOTE.** Conjecture 4 is still completely open.

The validity of this conjecture implies that the function $f : \mathbb{N} \to \mathbb{Z}^+$,

$$f(n) = \begin{cases} 1, & \text{if} \quad n = 0, \\ 1, & \text{if} \quad n = 1. \\ p_{n-1}!, & \text{if} \quad n \geq 2. \end{cases}$$

is an abstract factorial, see [13]. Thus, if true, it would follow from the results in [13] that for any abstract factorial $n!_a$, the quantity $\sum_{n\geq 1} 1/n!_a f(n) \notin \mathbb{Q}$.

## Apéry numbers

We define the Apéry numbers $A_n, B_n$ recursively, as usual, by setting $A_0 = 1, A_1 = 5$; $B_0 = 0, B_1 = 6$ whose general tems are given by the recurrence relations

$$A_{n+1} = (P(n)A_n - n^3 A_{n-1})/(n+1)^3,$$

and

$$B_{n+1} = (P(n)B_n - n^3 B_{n-1})/(n+1)^3,$$

where $P(n)$ is the polynomial

$$P(n) = 34n^3 + 51n^2 + 27n + 5.$$

In a singular argument Apéry [1] showed that $B_n/A_n \to \zeta(3)$ as $n \to \infty$ where $\zeta$ is the usual Riemann zeta function. In addition, he proved that $\zeta(3)$ is irrational (though no explicit formula akin to the one known for the values of $\zeta$ at positive even integers was given). More explicit proofs appeared since, e.g., [2], [15], [10] among others. (See [14] for extensions of the series acceleration method found in [Fischler [12], Remarque 1.3] to integer powers of $\zeta(3)$.)

Here we propose using an old irrationality criterion due to Brun [6] (see also [7]) in order to formulate a conjecture that, if true, would give another proof of the irrationality of $\zeta(3)$. Let $x_n$ be a sequence of real numbers and $\Delta$ the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$.

**Theorem 0.2.** *(Brun, [6]) Let $x_n \in \mathbb{Z}^+$ be an increasing sequence and $y_n \in \mathbb{Z}^+$ be such that $\Delta(y_n/x_n) > 0$. If*

$$\delta_n \equiv \Delta\left(\Delta y_n/\Delta x_n\right) < 0, \tag{4}$$

*then $y_n/x_n$ converges to an irrational number.*

Although Brun claimed later [7] that "...this theorem is simple but unfortunately not very useful" we show that perhaps it may be used to prove the irrationality of $\zeta(3)$.

The idea is as follows: It is known that the sequence $A_n$ of Apéry numbers is an increasing sequence of positive integers [10] and although the $B_n$ is not necessarily a sequence of integers, the weighted sequence $e_n B_n$ is such a sequence where $e_n = 2 \cdot (\text{lcm}\{1, 2, \ldots, n\})^3$, [10]. In addition, the sequence $B_n/A_n = e_n B_n/e_n A_n$ is increasing, [10] and it is easily proved that the sequence $e_n A_n$ is increasing as well.

Thus, setting $x_n = e_n A_n$ and $y_n = e_n B_n$ we see that the requirements $x_n$ is increasing and $y_n/x_n$ increasing are met in Theorem 0.2 (all sequences being positive and all integers). We anticipate the following

**Conjecture 5.** *There is an unbounded subsequence of positive integers $n_k \to \infty$ such that $\delta_{n_k} < 0$.*

**NOTE.** Conjecture 5 was proved in the affirmative by Lee Butler in [8].

Since it is known that $y_n/x_n$ increases to $\zeta(3)$, clearly $y_{n_k}/x_{n_k}$ does the same for any subsequence. Hence, an affirmative answer to the previous conjecture implies the irrationality of $\zeta(3)$ by Brun's irrationality theorem,Theorem 0.2. The numerical evidence seems to point to a stronger conjecture however. Indeed, it appears as if

**Conjecture 6.** *For every integer $N \geq 2$, there is an $n \in \mathbb{Z}^+$ such that all*

$$\delta_n, \delta_{n+1}, \delta_{n+2}, \ldots, \delta_{n+N} < 0.$$

**NOTE.** Conjecture 6 was proved in the affirmative by Lee Butler in [8].

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