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Formulas for $\pi(x)$ and the nth Prime

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Abstract

Using inequalities of Rosser and Schoenfeld, we complete the first author's partial proof of exact formulas for the prime-counting function $\pi(x)$ and the nth prime number p_n. Only the four arithmetic operations and the floor function are involved in the formulas. We indicate how to modify them in order to accelerate their computation.

S. Regimbal [2] and S. M. Ruiz [4, 5] have given formulas for the nth prime number p_n and the prime-counting function $\pi(x)$ which use only the elementary operations $+, -, \times, \div$ and the floor function $\lfloor \cdot \rfloor$. (See [1] for a survey of formulas for primes.) Regimbal's proof relies only on Bertrand's Postulate (between any number and its double there is always a prime), but his formula involves more than 2^n steps. Ruiz's formula requires only $O((n \log n)^3)$ steps, but his conditional proof assumes certain inequalities based on the Prime Number Theorem. In this note, we use inequalities of Rosser and Schoenfeld [3] to give a complete proof of the slightly modified formulas

$$\pi(x) = \sum_{j=2}^{\lfloor x \rfloor} \left(1 + \left\lfloor \frac{2 - \sum_{i=1}^{j} \left(\left\lfloor \frac{j}{i} \right\rfloor - \left\lfloor \frac{j-1}{i} \right\rfloor \right)}{j} \right\rfloor \right), \tag{1}$$

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$$p_n = 2 + \sum_{k=2}^{\lfloor 2n\log n+2\rfloor} \left(1 - \left\lfloor \frac{\pi(k)}{n} \right\rfloor\right), n > 1. \tag{2}$$

After the proof, we indicate various ways to modify and implement the formulas so that they operate in time $O(x^{3/2})$ and $O((n\log n)^{3/2})$, respectively.
Proof. For n a positive integer, let

$$d(n) := \sum_{d|n} 1 \tag{3}$$

denote the number of divisors of n. In [4], the first author found the formula

$$d(n) = \sum_{i=1}^{n} \left(\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor \right), \tag{4}$$

which holds since the quantity in parentheses is 1 or 0 according as i does or does not divide n.

Let F be the characteristic function of the set of prime numbers

$$F(n) := \begin{cases} 1 & if\ n\ is\ prime \\ 0 & otherwise. \end{cases} \tag{5}$$

From (3), we have $d(n) = 2$ if n is prime, and $d(n) > 2$ if n is composite. Since $2 \le d(n) \le n$ for $n > 1$, we have the formula

$$F(n) = 1 + \left\lfloor \frac{2 - d(n)}{n} \right\rfloor, n > 1. \tag{6}$$

Using (5), we write the function $\pi(x)$, defined as the number of primes not exceeding x, as the sum

$$\pi(x) = \sum_{j=2}^{\lfloor x \rfloor} F(j) \tag{7}$$

with the convention that any sum $\sum_{i=a}^{b}$ is zero if $a > b$. From (7), (6), (4), we obtain formula (1) for $\pi(x)$.

In order to derive formula (2) for p_n from (1), we will use the following lemma.
Lemma. *For $n > 1$, we have the inequalities*

$$\pi(2n\log n + 2) < 2n, \tag{8}$$

$$p_n < 2n \log n + 2. \tag{9}$$

Proof. Rosser and Schoenfeld [3] proved that

$$p_n > n \log n, n \in \mathbb{N}, \tag{10}$$

$$p_n < n \log n + n(\log \log n - 1/2), n > 20. \tag{11}$$

From (10), we have $p_{2n} > 2n \log 2n$. Since $\pi(p_{2n}) = 2n$, it follows that $\pi(2n \log 2n) < 2n$, which implies (8) if $n > 1$.

To prove (9) for $n > 1$, we verify it numerically for $n = 2, 3, ..., 20$, and note that (11) implies (9) for $n > 20$. This proves the lemma. □

For $n > 1$, the Lemma implies that

$$\left\lfloor \frac{\pi(k)}{n} \right\rfloor = \begin{cases} 0 & if \;\; 1 \le k \le p_n - 1 \\ 1 & if \;\; p_n \le k < 2n \log n + 2. \end{cases}$$

The desired formula for the nth prime number follows immediately. This completes the proof of (1) and (2). □

Optimizations. As they stand, the formulas for p_n, $\pi(x)$ and $d(n)$ operate in time $O((n \log n)^3)$, $O(x^2)$ and $O(n)$, respectively. We can improve these bounds by modifying the formulas as follows. If i divides n, so does n/i; thus to compute $d(n)$ it suffices to consider only $i \le \sqrt{n}$. This reduces the time for $d(n)$ (hence also for $F(n)$) to $O(n^{1/2})$, and for $\pi(x)$ to $O(x^{3/2})$. Computing $\pi(k)$ recursively as $\pi(k) = \pi(k-1) + F(k)$ for $k < 2n \log n + 2$ reduces the time for p_n to $O((n \log n)^{3/2})$.

We can also improve the computation time (but not the $O(.)$ bounds) in the following two ways. First, instead of the floor of n/i, use the integer quotient of n by i. Second (as P. Sebah [6] has pointed out), in formulas (2) and (7) for $\pi(x)$, after j= 2 we only need to sum over odd numbers, after j= 3 only over numbers relatively prime to 6, and similarly for other moduli 30, 210, 2310, . . ., m. This "sieving" cuts computation time by a factor of $m/\varphi(m)$, where φ is Euler's totient function.

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