A Collection of Number and Function Characterizations

Badih Ghusayni
Department of Mathematics
Lebanese University
Faculty of Science-1
LEBANON
bgou@ul.edu.lb http://www.ul.edu.lb/francais/publ/ghus.htm

Abstract : - The objective of this paper is to give characterizations of some numbers and many functions among which are the exponential functions, the identity function, the constant functions, the Gamma function, the trigonometric functions, and the hyperbolic trigonometric functions.

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1 Introduction

A **prime triplet** consists of three primes of the form N, N+2, N+4. It is quite easy to show that the only prime triplet is (3,5,7). A **twin prime** consists of two primes of the form N, N+2. In contrast, it is not known whether the number of twin primes is finite or infinite.
It is also easy to show that the only solution to $y^2 - x^3 = -1$ is (1,0). But probably the most striking and newly settled result which we can classify under this topic is Catalan's Conjecture proved by Preda Mihailescu [3] :

Theorem 1.1 The only solution to $x^u - y^v = 1$ in positive integers **x** and **y** and integers **u,v > 1** is $3^2 - 2^3 = 1$.

An interesting result due to Dickson [9] is the following theorem:

Theorem 1.2 All integers, except 23 and 239, are sums of eight cubes.

The following theorem is easy to prove:

Theorem 1.3

a) 3 is the only prime number of the form $N^2 - 1$, where N is a natural number

b) 5 is the only prime number of the form $N^2 - 4$.

Definition A natural number is called **perfect** if it is equal to the sum of its positive divisors, excluding itself.

The following is a nice result:

Theorem 1.4 [8, p.72]

28 is the only perfect number of the form $N^N + 1$.

We turn now to function characterizations:

Theorem1.5[4,p.230]

If $f(z) = \sum_{0}^{\infty} a_n z^n$ is a power series such that

i) $a_n > 0$ for n=0,1,2,...,

ii) $0 < r_n < R < \infty$, n = 0, 1, 2, ... where **R** is the radius of convergence and

iii) there exists a positive real number such that $a_1 = \alpha + 1$ and $f(r_n) = \left(\frac{n+\alpha}{\alpha}\right)^{\alpha+1}$ for n=0,1,2,...,

then

$f(z) = (1-z)^{-\alpha-1}$ for all $|z| < 1$.

2 Characterization of $f(z)=\frac{z}{(1-z)^2}$

Consider the power series

$$f(x)=x+2x^2+3x^3+...=\sum_1^{\infty} nx^n =$$

$$\frac{x}{(1-x)^2}, \text{with } -1<x<1.$$

Here $r_n=\frac{n}{n+1}$ for $n = 0, 1, 2, ...$

Since $0<r_n<1, f(r_n)=n(n+1), n=0,1,2,....$ This suggests the following:

Theorem 2.1 If $f(z)=\sum_1^{\infty} b_n z^n$ is a power series such that

i) $b_n>0$ for $n = 1, 2, ...$

ii) $0<s_n<R<\infty$, $n = 0, 1, 2, ...$ where R is the radius of convergence and $s_n=\frac{b_n}{b_{n+1}}$ and

iii) $b_1=1, b_2=2$, and $f(s_n)=n(n+1)$, then

$$f(z)=\frac{z}{(1-z)^2} \text{ for all } |z|<1.$$

Proof See [2]

3 Characterization of $f(z)=\frac{1+z}{(1-z)^2}$

Consider the power series

$$f(x)=1+3x+5x^2+...=\sum_0^{\infty}(2n+1)x^n$$

$$=\frac{1+x}{(1-x)^2},\ -1<x<1.$$

Here $r_0=0$ and $r_n=\frac{2n-1}{2n+1}$ for $n = 1, 2, ...$

Since $0\le r_n<1$ for $n = 0, 1, 2, ..., f(r_0)=1$, and $f(r_n)=n(2n+1)$, $n = 1, 2, ...$ This suggests the following

Theorem 3.1 If $f(z)=\sum_0^{\infty} a_n z^n$ is a power series such that

i) $a_n>0$ for $n = 0, 1, 2, ...$

ii) $0<r_n<R<\infty$, $n = 0, 1, 2, ...$ where R is the radius of convergence, $r_n=\frac{a_{n-1}}{a_n}$ and

iii) $a_0=1, a_1=3$, and $f(r_n)=n(2n+1)$, $n = 1, 2, ...$

then

$$f(z)=\frac{1+z}{(1-z)^2} \text{ for all } |z|<1$$

Proof See [2].

4 Characterizations of some exponential functions

Theorem 4.1 [4, p. 233] If $f(z)=\sum_0^{\infty} a_n z^n$ is an entire function such that

i) $a_n>0$ for $n = 0, 1, 2, ...,$ and

ii) $a_1=1, f(r_n)=e^n$ for $n = 0, 1, 1, 2, ...,$

then

$$f(z)=e^z \text{ for all } z.$$

The following is of the same type as the theorem above:

Theorem 4.2 [10, Theorem 9.3, p. 139] Suppose $f(z)$ is defined on the whole complex plane such that

i) it takes real values when z is real and, in particular, the value e when $z=1$.

ii) $f(z+w)=f(z)+f(w)$ for all complex numbers z and w, and

iii) $f(z)$ is entire,

then

$f(z)=e^z$.

Theorem 4.3 Suppose f is an analytic function on the disk D(0,R) such that for every z in the disk, $f(z)=f'(z)$ and $f(0)=1$. Then $f(z)=e^z$ on the whole complex plane.

Proof Put $f(z)=\sum_0^{\infty} a_n z^n$ on D(0,R).

Then $f'(z) = \sum_{0}^{\infty}(n+1)z^n$ on D(0,R).
Since $f(z) = f'(z)$, it follows that

$\frac{a_{n+1}}{a_n} = \frac{1}{n+1}$ which tends to 0 as **n** tends to ∞.
Using the Cauchy-Hadamard definition of the radius of convergence ρ, it follows that $\rho = \infty$. An easy induction proof shows that

$a_n = \frac{1}{n!}$ and so $f(z) = e^z$ on D(0,R).
Finally, the function e^z is entire and it is the analytic extension of **f** on the complex plane since D(0,R) contains a compact neighborhood containing infinitely many elements.

We now state some interesting results of the same type but this time on functions of a real variable:

Theorem 4.4 [1, p. 106]
Suppose that **f** is a real-valued function on the set of real numbers such that
i) **f(x+y)=f(x)f(y),**
ii) **f**(x) is differentiable at 0,
iii) **f** is not identically 0,
then
f is differentiable everywhere, $f'(x) = f'(0)f(x)$,
and
$f(x) = e^{f'(0)x}$.

Theorem 4.5
Suppose that **f** is a real-valued function such that **f(x+y)=f(x)f(y).** Then
a) **f**(1) ≥ 0,
b) If **f**(1) = 0, then **f(x) = 0** for all **x.**
c) If **f**(1) = a > 0, then **f(p)** = a^p for every **p** in the set of natural numbers N and $f(\frac{p}{q}) = a^{\frac{p}{q}}$ for every $\frac{p}{q}$ in the set of rational numbers **Q**.
d) If **f(1)=a > 0** and f is continuous at each real number, then $f(x) = a^x$ for all **x.**

We conclude this section by stating and proving the following

Theorem 4.6
Suppose **f(z)** is a function from the non-negative numbers into the non-negative numbers such that
i) $(-1)^p f^{(p)}(0) > 0$ for every positive integer **p.**
ii) $\int_0^{\infty} f(x)dx = 1$, and
iii) **f(0)=1,**
then
$f(x) = e^{-x}$.

Proof. Let $g(x) = f(x) - e^{-x}$. Then **g(0)=1.**
Moreover, $\int_0^{\infty} g(x)dx = \int_0^{\infty} f(x)dx - \int_0^{\infty} e^{-x}dx = 0$.
Using i) and the Maclaurin's expansion of **g(x)** it follows that $g(x) \geq 0$. Consequently, **g(x)=0.**

5 Characterization of f(x) = x

Theorem 5.1
Suppose **f** is a function from the non-negative integers into itself such that **f(1) > 0** and $f(m^2 + n^2) = [f(m)]^2 + [f(n)]^2$. Then **f** is the identity function.

Proof See [1, pp. 106,107].

6 Characterization of Constant functions

Theorem 6.1 Let **f** be an entire function. Then **f** is a constant function if and only if one of the following conditions is satisfied,
a) Re(**f**) > 1,
b) There is $\in$ **> 0** and w $\in$ C such that |**f(z)**-w| > $\in$ for all z $\in$ C,
c) **f(z)=f(z+1)** and **f(z)=f(z+i).**

Proof It is clear that a constant function is an entire function which satisfies the given three conditions. So we will show that each of the given conditions applied with an entire function **f** imply that **f** is a constant function.
a) Since Re(**f**) > 1, |**f(z)**| > 1 and in particular $f(z) \neq 0$ and therefore by letting
$g(z) = \frac{1}{f(z)}$, **g(z)** is an entire function with

$|g(z)| < 1$. By Liouville's Theorem, **g** and hence **f** is a constant function.

b) Since $|f(z)-w| > \epsilon$, $|f(z)-w| \neq 0$. Thus $h(z) = \frac{1}{|f(z) - w|}$ is a bounded entire function and so it must be a constant and therefore $f(z) = w + \frac{1}{c}$ is constant.

c) Consider the square S with vertices (0,0), (1,0), (1,1), and (0,1). Since f is in a continuous function on the compact set S, it must be bounded on S. Thus there is a positive constant M such that $|f(z)| < M$ for all $z \in S$.

Then we can find integers **k** and **n** such that $|f(z)|=|f(z\pm 1)|=\ldots=|f(z\pm k)|=$ $|f(z+k\pm i)|=|f(z+k\pm 2i)|=\ldots=|f(z+k \pm ni)|< M$. Since $z+k\pm ni \in S$, **f** is bounded and since it is entire, it must be a constant.

7 Characterization of the zero function

We list without proof the following results:

Theorem 7.1 (Carlson, consult [1] for definitions) Let **f(z) be** holomorphic and of exponential type in the half plane $x \geq 0$ and let its **indicator function** $h(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r}$ satisfy $h(\frac{\pi}{2}) + h(-\frac{\pi}{2}) < 2\pi$. If $f(n) = 0, n = 0,1,2,...,$ then $f(z)=0$.

Theorem 7.2 Let $f(z) = \sum_{1}^{\infty} a_n z^n$ have radius of convergence R>0. If $f(\frac{1}{n}) = 0$ for $n = 0,1,2,...$, then **f(z)=0.**

Theorem 7.3 Let f be a continuous function on **[a,b]** and suppose that $\int_a^b x^n f(x)dx = 0$ for $n = 0,1,2,...$ Then **f=0.**

Theorem 7.4 Let f be a continuous function on $[-\pi, \pi]$ and suppose that $\int_{-\pi}^{\pi} f(x)\cos(nt)dx = 0$ or $\int_{-\pi}^{\pi} f(x)\sin(nt)dx = 0$ for $n = 0,1,2,...$ Then f=0.

8 Characterization of $f(x) = \Gamma(x)$

Theorem 8.1 Suppose **f(x)** is a real-valued function such that

i) **f(x)** is twice differentiable on $(0, \infty)$,
ii) **f(x) > 0,**
iii) **f(1)=1,**
iv) **f(x+1)=xf(x),** and
v) $f''(x)f(x) - [f'(x)]^2 > 0$,

then $f(x) = \Gamma(x)$ for all $x > 0$.

Proof (see [1,p. 108] for a relatively easy proof)

9 Characterizations of other Functions

Theorem 9.1 Suppose that f is a twice-differentiable function from the real numbers into itself such that $f(x + y)f(x - y) = [f(x)]^2 - [f(y)]^2$. Then **f(x)=mx, f(x)=A sin (ax)** or **f(x)=c sinh (kx),** for some constants m, A, a, c and k.

Proof See [7, p. 640].

It is easy to prove the following results:

Theorem 9.2 Suppose that f is a function defined on the real numbers such that $f(x + y) = f(x) + f(y)$. Then

a) For each $r \in Q$, **f(r)=f(1)r.**
b) If f is continuous at 0, then f is continuous at all $x \in R$.
c) If f is continuous on R, then $f(\lambda x) = \lambda f(x)$ for every $\lambda \in R$.
d) If f is a linear function from R into R, then $f(x) = \alpha x$ for some $\alpha \in R$.

Theorem 9.3 Let $f(z)=\sum_{1}^{\infty} a_n z^n$ have radius of convergence R>0.

a) If $f(\frac{1}{n})=\frac{2}{n}$ for $n = 0,1,2,\ldots$, then $f(z)=2z$.

b) If $f(\frac{1}{n})=\frac{n}{n+1}$ for $n = 0,1,2,\ldots$, then

$$f(z)=\frac{1}{1+z}, |z|<1.$$

Theorem 9.4 Suppose that $P(x)$ is a second-degree polynomial such that $P(0)=1$, $P'(0)=0$, and $\int \frac{P(x)}{x^3(x-1)^2}dx$ is a rational function. Then $P(x)=-3x^2+1$.

Theorem 9.5 Suppose that $f(z)$ is an entire function such that $|f'(z)|<|z|$, for all z. Then $f(z)=a+bz^2$ with $|b|<\frac{1}{2}$.

We conclude this section by stating and proving the following theorems:

Theorem 9.6 If $f(z)=\sum_{0}^{\infty} a_n z^n$ is a power series such that $f''(z)+\lambda^2 f(z)=0, f(0)=0, f'(0)=\lambda$, a complex constant, then $f(z)=\lambda\sin(\lambda z)$.

Proof $f''(z)=\sum_{0}^{\infty}(n+2)(n+1)c_{n+2}z^n$. It follows that $c_n=0$ for even n, and $c_{2n+1}=\frac{-\lambda^2 c_{2n-1}}{(2n+1)2n}$.

Proceeding we get $c_{2n+1}=\frac{(-1)^{\lambda}\lambda^{2n+1}}{(2n+1)!}$.

Consequently, $f(z)=\lambda\sin(\lambda z)$.

Theorem 9.7 If $f(z)$ is an entire function such that

a) $z=0$ is a zero of $f(z)$ of order m,

b) $|f^{(m-1)}(z)|<K|z|$ for every $z\in C$, and

c) $f(i)=-2$,

then

$$f(z)=-\frac{2z^m}{i^m}.$$

Proof By a) $z=0$ is a simple zero of $f^{(m-1)}(z)$..

So $z=0$ is a removable singularity of $\frac{f^{(m-1)}(z)}{z}$.

Using b) $\frac{f^{(m-1)}(z)}{z}$ is entire and bounded and therefore must be constant by Liouville's Theorem. That is, there is $\lambda\in C$ such that $f^{(m-1)}(z)=\lambda z$. Taking antiderivative and a) into account

$$f(z)=-\frac{\lambda z^m}{m!}.$$

By c) $\lambda=\frac{-2(m!)}{i^m}$.

Consequently, $f(z)=-\frac{2z^m}{i^m}$.

10 Concluding Remarks for Future Research

This was just a relatively large sample of characterizations. Needless to say, that opens the door for further explorations in many directions: Exploring other numbers, adding different hypotheses to the theorems on functions that appeared in this article and exploring other functions.

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