

CHARACTERIZATIONS OF ARITHMETICAL PROGRESSION  
SERIES WITH SOME COUNTEREXAMPLES  
ON INTERPOLATION

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**Abstract.** Characterizations of the functions

$$f(z) = \frac{z}{(1-z)^2} \text{ and } f(z) = \frac{1+z}{(1-z)^2}$$

are given. We also give counterexamples to show that some generalized problems on interpolation do not hold.

**1. Introduction.** Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a power series with positive coefficients and positive radius of convergence. Associate  $a_{-1} = 0$ ,  $r_n = a_{n-1}/a_n$  for  $n = 0, 1, 2, \dots$ . It is known [3] that if

$$\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = R,$$

then the radius of convergence of

$$\sum_{n=1}^{\infty} a_n z^n$$

is  $R$  and also [2] if

$$\liminf_{n \rightarrow \infty} \frac{\log(\frac{a_n}{a_{n+1}})}{\log n} = L > 0,$$

then

$$\sum_{n=0}^{\infty} a_n z^n$$

is an entire function of order  $\leq 1/L$ . In addition, if

$$\lim_{n \rightarrow \infty} \frac{\log\left(\frac{a_n}{a_{n+1}}\right)}{\log n} = L > 0,$$

then

$$\sum_{n=0}^{\infty} a_n z^n$$

is of order  $1/L$ .

In his paper [1], Abi-Khuzam used the term “normalize” as follows. If  $g(z) = f(cz)$  with  $c$  a positive constant, then  $g(z)$  will have positive coefficients  $\{b_n\}$  and if  $s_n = b_{n-1}/b_n$ , then  $g(s_n) = f(r_n)$ ,  $n = 0, 1, 2, \dots$ . Thus, one can normalize the function  $f(z)$  to make  $a_1$  equal to a given number without changing the sequence  $\{f(r_n)\}$  and such a normalization was incorporated in [1] where the following theorems were proven.

Theorem 1 [1]. If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a power series such that

- (i)  $a_n > 0$  for  $n = 0, 1, 2, \dots$ ,
- (ii)  $0 < r_n < R < \infty$ , where  $R$  is the radius of convergence, and
- (iii) there exists a positive real number  $\alpha$  such that  $a_1 = \alpha + 1$  and

$$f(r_n) = \left(\frac{n + \alpha}{\alpha}\right)^{\alpha+1} \quad \text{for } n = 0, 1, 2, \dots,$$

then

$$f(z) = (1 - z)^{-\alpha-1} \quad \text{for all } |z| < 1.$$

Theorem 2 [1]. If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is an entire function such that

- (i)  $a_n > 0$  for  $n = 0, 1, 2, \dots$ , and
- (ii)  $a_1 = 1$  and  $f(r_n) = e^{r_n}$  for  $n = 0, 1, 2, \dots$ ,

then

$$f(z) = e^z \text{ for all } z.$$

In concluding an interesting paper, Abi-Khuzam [1] raised the following questions.

Problem 1. If

$$f(r) = \sum_{n=0}^{\infty} a_n r^n, \quad g(r) = \sum_{n=0}^{\infty} b_n r^n,$$

$r_n = a_{n-1}/a_n$ ,  $s_n = b_{n-1}/b_n$ , and  $f(r_n) = g(s_n)$  for all  $n \geq 0$ , does it follow that modulo normalization  $f = g$ ?

Problem 2. Since the hypothesis of Theorem 2 holds for functions with positive and negative coefficients e.g.,  $f(z) = e^{-z}$ , one may ask whether it holds true under the assumption  $a_n \neq 0$ , instead of  $a_n > 0$ .

In this paper we will find characterizations of a type similar to Theorems 1 and 2 and also solve Problems 1 and 2.

**1. Characterization of  $f(z) = z/(1-z)^2$ .** Consider the power series

$$f(x) = x + 2x^2 + 3x^3 + \dots = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}, \quad -1 < x < 1.$$

Here  $r_n = n/n + 1$  for  $n = 0, 1, 2, \dots$ . Since  $0 \leq r_n < 1$ ,  $f(r_n) = n(n+1)$ ,  $n = 0, 1, 2, \dots$ . This suggests the following theorem.

Theorem 3. If

$$f(z) = \sum_{n=1}^{\infty} b_n z^n$$

is a power series such that

- (i)  $b_n > 0$  for  $n = 1, 2, \dots$
- (ii)  $0 \leq s_n < R < \infty$ ,  $n = 0, 1, 2, \dots$ , where  $R$  is the radius of convergence and  $s_n = b_n/b_{n+1}$ , and
- (iii)  $b_1 = 1$ ,  $b_2 = 2$ , and  $f(s_n) = n(n+1)$ ,

then

$$f(z) = \frac{z}{(1-z)^2} \text{ for all } |z| < 1.$$

Proof. It is clear that

$$f(z) = \frac{z}{(1-z)^2}$$

satisfies (i)–(iii). We now prove that

$$\frac{z}{(1-z)^2}$$

is the only such function. This is clearly true if  $z = 0$ . Suppose now that  $z \neq 0$ . Let

$$P(z) = \frac{f(z)}{z}.$$

Then

$$P(z) = \sum_{n=0}^{\infty} b_{n+1}z^n$$

with the same sequence  $\{s_n\}$  associated. Using  $b_1 = 1$  and  $b_2 = 2$  we can write

$$P(z) = 1 + 2z + \sum_{n=2}^{\infty} b_{n+1}z^n.$$

Letting  $a_n = b_{n+1}$ ,  $n = 0, 1, 2, \dots$  we have

$$P(z) = 1 + 2z + \sum_{n=2}^{\infty} a_n z^n$$

with the sequence

$$\{r_n\} = \left\{ \frac{a_{n-1}}{a_n} \right\}$$

associated. However,  $P(r_n) = P(s_n)$  for  $n = 0, 1, 2, \dots$ . Then

$$P(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } a_n > 0 \text{ for } n = 0, 1, 2, \dots,$$

$$0 < r_n = \frac{a_{n-1}}{a_n} = \frac{b_n}{b_{n+1}} = s_n < \infty, \quad a_1 = 2, \text{ and}$$

$$P(r_n) = P(s_n) = \frac{f(s_n)}{s_n}.$$

Next, we show by induction that  $s_n = n/n + 1$ . From the representation of  $P(z)$  above,  $s_1 = 1/2$ . Suppose then that  $s_n = n/n + 1$ . We shall show that  $s_{n+1} = (n+1)/(n+2)$ . Now  $s_1 s_2 \dots s_n = 1/b_{n+1}$ . Thus,

$$b_{n+2} = \frac{1}{s_1 s_2 \dots s_n s_{n+1}} = \frac{1}{\frac{1}{2} \frac{2}{3} \dots \frac{n}{n+1} s_{n+1}}.$$

Therefore,

$$b_{n+2} = \frac{n+1}{s_{n+1}} \text{ or } s_{n+1} = \frac{n+1}{b_{n+2}} = \frac{b_{n+1}}{b_{n+2}}$$

which implies that  $b_{n+1} = n+1$  and consequently  $s_{n+1} = (n+1)/(n+2)$ .

As a result we get

$$P(r_n) = \frac{n(n+1)}{\frac{n}{n+1}} = (n+1)^2$$

for  $n = 0, 1, 2, \dots$ . Thus, by Theorem 1 (with  $\alpha = 1$ ) it follows that

$$P(z) = \frac{1}{(1-z)^2}.$$

Consequently,

$$f(z) = \frac{z}{(1-z)^2}$$

and the proof of Theorem 3 is complete.

**3. Characterization of  $f(z) = (1+z)/(1-z)^2$ .** To find such a characterization we will use the following.

Lemma. The function

$$f(x) = \left(\frac{x}{x-1}\right)^{2x-1}$$

is a decreasing function on  $[2, \infty)$ .

Proof.  $\log f(x) = (2x-1)[\log x - \log(x-1)]$  implies

$$\frac{f'(x)}{f(x)} = 2 \log \frac{x}{x-1} - \frac{2x-1}{x(x-1)}$$

and it suffices to show that

$$\log \frac{x}{x-1} \leq \frac{2x-1}{2x(x-1)}.$$

But this follows easily using the derivative of

$$\log \frac{x}{x-1} - \frac{2x-1}{2x(x-1)}$$

and the proof of the lemma is complete.

Next consider the power series

$$f(x) = 1 + 3x + 5x^2 + \cdots = \sum_{n=0}^{\infty} (2n+1)x^n = \frac{1+x}{(1-x)^2}, \quad -1 < x < 1.$$

Here  $r_0 = 0$  and  $r_n = (2n - 1)/(2n + 1)$  for  $n = 1, 2, \dots$ . Since  $0 \leq r_n < 1$  for  $n = 0, 1, 2, \dots$ ,  $f(r_0) = 1$ , and  $f(r_n) = n(2n + 1)$ ,  $n = 1, 2, \dots$ . This suggests the following theorem.

Theorem 4. If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a power series such that

- (i)  $a_n > 0$  for  $n = 0, 1, 2, \dots$
- (ii)  $0 < r_n < R < \infty$ ,  $n = 0, 1, 2, \dots$ , where  $R$  is the radius of convergence and  $r_n = a_{n-1}/a_n$ , and
- (iii)  $a_0 = 1$ ,  $a_1 = 3$ , and  $f(r_n) = n(2n + 1)$ ,  $n = 1, 2, \dots$

then

$$f(z) = \frac{1+z}{(1-z)^2} \text{ for all } |z| < 1.$$

Proof. Clearly

$$\frac{1+z}{(1-z)^2}$$

satisfies (i)–(iii). We now prove that

$$\frac{1+z}{(1-z)^2}$$

is the only such function. Using  $a_0 = 1$  and  $a_1 = 3$  we can write

$$f(z) = 1 + 3z + \sum_{n=2}^{\infty} a_n z^n$$

with the sequence

$$\{r_n\} = \left\{ \frac{a_{n-1}}{a_n} \right\}$$

associated. Now  $f(r_1) = 3$ . Since  $r_1 = 1/3$ , we have  $3 = f(1/3)$ . The positivity of  $\{a_n\}$  implies  $f$  is increasing on  $[0, R)$ . By (iii),  $\{f(r_n)\}$  is increasing. So  $\{r_n\}$  is increasing. Thus,  $r_1 \leq r_n < R$  or

$$\frac{1}{3} \leq r_n < R \text{ for } n = 1, 2, \dots$$

Consequently,

$$\lim_{n \rightarrow \infty} r_n = R.$$

Let

$$\beta = \inf_{n \geq 1} \frac{2n+1}{2n-1} r_n \text{ and } \delta = \sup_{n \geq 1} \frac{2n+1}{2n-1} r_n.$$

Clearly

$$\frac{1}{3} \leq \beta \leq R \leq \delta \leq 3R.$$

Moreover,

$$\frac{2n+1}{\delta} r_n \leq 2n-1 \leq \frac{2n+1}{\beta} r_n, \text{ for } n = 1, 2, \dots$$

In particular ( $n = 1$ ) we see that  $\beta \leq 1$  and  $\delta \geq 1$ . Let  $r$  be a number such that  $0 < r < \beta$ . We shall show by induction that

$$f^{(n)}(r) \leq \frac{(2n+1 + \frac{\beta}{r}) \cdots (5 + \frac{\beta}{r})(3 + \frac{\beta}{r})}{2^n(\beta - r)^n} f(r), \text{ for } n = 1, 2, \dots$$

First

$$\begin{aligned} f'(r) &= \sum_{n=1}^{\infty} n a_n r^{n-1} = \frac{1}{2} \sum_{n=1}^{\infty} 2n a_n r^{n-1} = \frac{1}{2} \sum_{n=1}^{\infty} (2n-1) a_n r^{n-1} + \frac{1}{2} \sum_{n=1}^{\infty} a_n r^{n-1} \\ &\leq \frac{1}{2\beta} \sum_{n=1}^{\infty} (2n+1) a_{n-1} r^{n-1} + \frac{1}{2} \frac{f(r)}{r} = \frac{1}{2\beta} \sum_{n=0}^{\infty} (2n+3) a_n r^n + \frac{1}{2} \frac{f(r)}{r} \\ &= \frac{1}{\beta} \sum_{n=0}^{\infty} n a_n r^n + \left(\frac{3}{2\beta} + \frac{1}{2r}\right) f(r) = \frac{r}{\beta} f'(r) + \left(\frac{3}{2\beta} + \frac{1}{2r}\right) f(r). \end{aligned}$$



Thus,

$$f'(r) \leq \frac{1}{2} \frac{3 + \frac{\beta}{r}}{\beta - r} f(r). \quad (1)$$

Assume the induction hypothesis that

$$\begin{aligned} f^{(k-1)}(r) &= \sum_{n=k-1}^{\infty} n(n-1) \cdots (n-k+2) a_n r^{n-k+1} \\ &\leq \frac{(2k-1 + \frac{\beta}{r}) \cdots (5 + \frac{\beta}{r})(3 + \frac{\beta}{r})}{2^{k-1}(\beta - r)^{k-1}} f(r) \end{aligned}$$

for some integer  $k \geq 2$ . Then

$$\begin{aligned} f^{(k)}(r) &= \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n r^{n-k} \\ &= \sum_{n=1}^{\infty} (n+k-1)(n+k-2) \cdots n a_{n+k-1} r^{n-1} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} (2n+2k-2)(n+k-2) \cdots n a_{n+k-1} r^{n-1} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} n \cdots (n+k-2) [2(n+k-1) - 1] a_{n+k-1} r^{n-1} \\ &\quad + \frac{1}{2} \sum_{n=1}^{\infty} n \cdots (n+k-2) a_{n+k-1} r^{n-1} \\ &\leq \frac{1}{2\beta} \sum_{n=1}^{\infty} n \cdots (n+k-2) [2(n+k-1) + 1] a_{n+k-2} r^{n-1} \\ &\quad + \frac{1}{2} \sum_{n=1}^{\infty} n \cdots (n+k-2) a_{n+k-1} r^{n-1} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\beta} \sum_{n=1}^{\infty} n \cdots (n+k-2) \left[ (n+k-1) + \frac{1}{2} \right] a_{n+k-2} r^{n-1} \\
&\quad + \frac{1}{2} \sum_{n=1}^{\infty} n \cdots (n+k-2) a_{n+k-1} r^{n-1} \\
&\leq \frac{1}{\beta} \sum_{n=1}^{\infty} n \cdots (n+k-2) n a_{n+k-2} r^{n-1} \\
&\quad + \left( k - \frac{1}{2} \right) \frac{1}{\beta} \sum_{n=1}^{\infty} n \cdots (n+k-2) a_{n+k-2} r^{n-1} + \frac{1}{2} \frac{f^{(k-1)}(r)}{r} \\
&= \frac{1}{\beta} \sum_{n=2}^{\infty} (n-1) n \cdots (n+k-2) a_{n+k-2} r^{n-1} \\
&\quad + \left( k + \frac{1}{2} \right) \frac{1}{\beta} \sum_{n=1}^{\infty} n \cdots (n+k-2) a_{n+k-2} r^{n-1} + \frac{1}{2} \frac{f^{(k-1)}(r)}{r} \\
&= \frac{1}{\beta} \sum_{n=1}^{\infty} n \cdots (n+k-1) a_{n+k-1} r^n + \left( k + \frac{1}{2} \right) \frac{1}{\beta} f^{(k-1)}(r) + \frac{1}{2} \frac{f^{(k-1)}(r)}{r} \\
&= \frac{r}{\beta} f^{(k)}(r) + \left( k + \frac{1}{2} \right) \frac{1}{\beta} f^{(k-1)}(r) + \frac{1}{2} f^{(k-1)}(r) r \\
&\frac{\beta-r}{\beta} f^{(k)}(r) \leq \left( k + \frac{1}{2} \right) \frac{1}{\beta} f^{(k-1)}(r) + \frac{1}{2} \frac{f^{(k-1)}(r)}{r}.
\end{aligned}$$

Thus,

$$\begin{aligned}
f^{(k)}(r) &\leq \frac{1}{2} \frac{2k+1}{\beta-r} f^{(k-1)}(r) + \frac{1}{2} \frac{\beta}{r} \frac{1}{\beta-r} f^{(k-1)}(r) \\
&= \frac{1}{2(\beta-r)} \left[ 2k+1 + \frac{\beta}{r} \right] f^{(k-1)}(r).
\end{aligned}$$

Consequently,

$$f^{(k)}(r) \leq \frac{1}{2^k} \frac{(2k+1+\frac{\beta}{r})(2k-1+\frac{\beta}{r}) \cdots (5+\frac{\beta}{r})(3+\frac{\beta}{r})}{(\beta-r)^k} f(r)$$

which completes the induction proof.

We now use inequality (1), Taylor's Theorem, and the binomial series to obtain for  $0 < c \leq r < \beta$

$$\begin{aligned} f(r) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (r-c)^n \\ &\leq f(c) \sum_{n=0}^{\infty} \frac{(2n+1+\frac{\beta}{r}) \cdots (5+\frac{\beta}{r})(3+\frac{\beta}{r})}{2^n n! (\beta-c)^n} (r-c)^n \\ &\leq f(c) \left[ 1 + \sum_{n=1}^{\infty} \frac{(2n+1+\frac{\beta}{r}) \cdots (5+\frac{\beta}{r})(3+\frac{\beta}{r})}{2^n n!} \left( \frac{r-c}{\beta-c} \right)^n \right] \\ &= f(c) \left( \frac{\beta-c}{\beta-r} \right)^{\frac{3+\frac{\beta}{r}}{2}}. \end{aligned}$$

Next, let  $s$  be a number such that  $0 \leq s < \beta$ . We shall show that

$$f^{(n)}(s) \geq \frac{(2n+1) \cdots 5 \cdot 3}{2^n (\delta-s)^n} f(s) \text{ for } n = 1, 2, \dots$$

Now,

$$\begin{aligned} f'(s) &= \sum_{n=1}^{\infty} n a_n s^{n-1} = \frac{1}{2} \sum_{n=1}^{\infty} 2n a_n s^{n-1} \\ &\geq \frac{1}{2} \sum_{n=1}^{\infty} (2n-1) a_n s^{n-1} \geq \frac{1}{2\delta} \sum_{n=1}^{\infty} (2n+1) a_{n-1} s^{n-1} \\ &= \frac{1}{2\delta} \sum_{n=0}^{\infty} (2n+3) a_n s^n = \frac{s}{\delta} f'(s) + \frac{3}{2\delta} f(s). \end{aligned}$$

Thus,

$$f'(s) \geq \frac{3}{2(\delta - s)} f(s). \quad (2)$$

Assume the induction hypothesis that

$$\begin{aligned} f^{(k-1)}(s) &= \sum_{n=k-1}^{\infty} n(n-1)\cdots(n-k+2)a_n r^{n-k+1} \\ &\geq \frac{(2k-1)\cdots 5\cdot 3}{2^{k-1}(\delta-s)^{k-1}} f(s) \end{aligned}$$

for some integer  $k \geq 2$ . Then,

$$\begin{aligned} f^{(k)}(s) &= \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n s^{n-k} \\ &= \sum_{n=1}^{\infty} (n+k-1)(n+k-2)\cdots n a_{n+k-1} s^{n-1} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} (2n+2k-2)(n+k-2)\cdots n a_{n+k-1} s^{n-1} \\ &\geq \frac{1}{2} \sum_{n=1}^{\infty} n\cdots(n+k-2)[2(n+k-1)-1] a_{n+k-1} s^{n-1} \\ &\geq \frac{1}{2\delta} \sum_{n=1}^{\infty} n\cdots(n+k-2)(2n+2k-1) a_{n+k-2} s^{n-1} \\ &= \frac{1}{2\delta} \sum_{n=0}^{\infty} (n+1)\cdots(n+k-1)(n+2k+1) a_{n+k-1} s^n \end{aligned}$$

$$\begin{aligned}
&= \frac{s}{\delta} \sum_{n=1}^{\infty} n(n+1) \cdots (n+k-1) a_{n+k-1} s^{n-1} \\
&\quad + \frac{2k+1}{2\delta} \sum_{n=0}^{\infty} (n+k-1) \cdots (n+2)(n+1) a_{n+k-1} s^n \\
&= \frac{s}{\delta} f^{(k)}(s) + \frac{2k+1}{2\delta} f^{(k-1)}(s).
\end{aligned}$$

Thus,

$$f^{(k)}(s) \geq \frac{1}{2} \frac{2k+1}{\delta-s} f^{(k-1)}(s)$$

and therefore (using the induction hypothesis)

$$f^{(k)}(s) \geq \frac{(2k+1) \cdots 5 \cdot 3}{2^k (\delta-s)^k} f(s)$$

which completes the induction proof.

Similarly we use inequality (2), Taylor's Theorem, and the binomial series to obtain for  $0 \leq c \leq s < \beta < \delta$ ,

$$f(s) \geq f(c) \left[ 1 + \sum_{n=1}^{\infty} \frac{(2n+1) \cdots 5 \cdot 3}{2^n n!} \left( \frac{s-c}{\delta-c} \right)^n \right] = f(c) \left[ \frac{\delta-c}{\delta-s} \right]^{\frac{3}{2}}.$$

Case 1. ( $\beta$  is achieved). There exists  $m \geq 1$  such that

$$\beta = \frac{2m+1}{2m-1} r_m.$$

We shall show that  $\beta = 1$ . If  $m = 1$  there is nothing to prove. Assume  $m \geq 2$ . Now  $\beta > r_m$  and we can use  $c = r_{m-1}$  and  $r = r_m$  to get

$$\left[ \frac{\beta - r_{m-1}}{\beta - r_m} \right]^{\frac{3 + \frac{\beta}{r_m}}{2}} \geq \frac{f(r_m)}{f(r_{m-1})}.$$

But,

$$\frac{\beta}{r_m} = \frac{2m + 1}{2m - 1}.$$

Therefore,

$$\left[ \frac{\beta - r_{m-1}}{\beta - r_m} \right]^{4m-1} \geq \left[ \frac{m(2m + 1)}{(m - 1)(2m - 1)} \right]^{2m-1}.$$

Now using the lemma we have

$$\left( \frac{m}{m - 1} \right)^{2m-1} \geq \left( \frac{m + \frac{1}{2}}{m - \frac{1}{2}} \right)^{2m}.$$

Thus,

$$\left[ \frac{\beta - r_{m-1}}{\beta - r_m} \right]^{4m-1} \geq \left[ \frac{2m + 1}{2m - 1} \right]^{2m} \left[ \frac{2m + 1}{2m - 1} \right]^{2m-1} = \left[ \frac{2m + 1}{2m - 1} \right]^{4m-1}$$

and consequently,

$$\frac{\beta - r_{m-1}}{\beta - r_m} \geq \frac{2m + 1}{2m - 1}.$$

So

$$\beta - r_{m-1} \geq \frac{2m + 1}{2m - 1} \beta - \beta.$$

It follows that

$$\beta \geq \frac{2m-1}{2m-3}\beta.$$

Since  $m-1 \geq 1$ ,

$$\beta \leq \frac{2m-1}{2m-3}r_{m-1}.$$

Therefore,

$$\beta = \frac{2m-1}{2m-3}r_{m-1}.$$

Proceeding as above it follows that  $\beta = 1$ .

Now by the definition of  $\beta$  we have

$$\frac{1}{r_n} \leq \frac{2n+1}{2n-1} \text{ for } n = 1, 2, \dots.$$

But,

$$a_n = \frac{1}{r_1 r_2 \dots r_n}$$

and hence,  $a_n \leq 2n+1$ . Thus,

$$3 = f\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{3}\right)^n \leq \sum_{n=0}^{\infty} (2n+1) \left(\frac{1}{3}\right)^n = 3$$

and so  $a_n = 2n+1$ . Consequently,

$$f(z) = \frac{1+z}{(1-z)^2} \text{ for all } |z| < 1$$

in this case.

Case 2. ( $\beta$  and  $\delta$  are not achieved). Here for every  $n = 1, 2, \dots$

$$\beta < \frac{2n+1}{2n-1}r_n < \delta$$

Since  $\beta$  is an infimum, a subsequence of

$$\left\{ \frac{2n+1}{2n-1}r_n \right\}$$

must converge to  $\beta$ . But as the subsequence itself is convergent we must have

$$\lim_{n \rightarrow \infty} \frac{2n+1}{2n-1}r_n = \beta.$$

Then

$$\lim_{n \rightarrow \infty} r_n = \beta.$$

So  $\beta = R$ . Since  $\delta$  is a supremum, we must also have

$$\lim_{n \rightarrow \infty} r_n = \delta.$$

Thus,  $\beta = R = \delta$  and using the definitions of  $\beta$  and  $\delta$  we have

$$\frac{2n+1}{2n-1}r_n = R$$

for all  $n = 1, 2, \dots$ . That is, this case cannot occur.

Case 3. ( $\beta$  is not achieved but  $\delta$  is). Here for every  $n = 1, 2, \dots$  we have

$$\beta < \frac{2n+1}{2n-1}r_n$$



and there is an integer  $m \geq 1$  such that

$$\delta = \frac{2m+1}{2m-1} r_m.$$

If  $m = 1$ , there is nothing to prove. Assume  $m \geq 2$ . As in Case 2,

$$\lim_{n \rightarrow \infty} \frac{2n+1}{2n-1} r_n = \beta$$

so that  $\beta = R$ . In particular,  $r_n < \beta < \delta$  for  $n = 0, 1, \dots$  and we can use  $c = r_{m-1}$  and  $s = r_m$  to get

$$\left( \frac{\delta - r_{m-1}}{\delta - r_m} \right)^3 \leq \left[ \frac{m(2m+1)}{(m-1)(2m-1)} \right]^2.$$

But

$$\frac{m}{m-1} \leq \left( \frac{2m+1}{2m-1} \right)^2.$$

So

$$\frac{\delta - r_{m-1}}{\delta - r_m} \leq \frac{2m+1}{2m-1}$$

and the result follows as in Case 1.

**4. Counterexamples.** Here is a counterexample for problem 1.

Let

$$f(r) = \sum_{n=0}^{\infty} \frac{1}{e^n(n+1)^2} r^n$$

and let

$$g(r) = \sum_{n=0}^{\infty} \frac{e^n}{(n+1)^2} r^n.$$

Clearly,

$$r_n = e\left(1 + \frac{1}{n}\right)^2 \text{ and } s_n = \frac{1}{e}\left(1 + \frac{1}{n}\right)^2 \text{ for } n \geq 1.$$

Now

$$f(r_k) = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \left[ \left(1 + \frac{1}{k}\right)^n \right]^2 = g(s_k) \text{ for } k \geq 1.$$

Moreover, since  $r_0 = 0$  and  $s_0 = 0$ , it follows that  $f(r_0) = 1 = g(s_0)$  and therefore  $f(r_k) = g(s_k)$  for  $k \geq 0$ . However,  $f \neq g$ .

Here is a counterexample for Problem 2.

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function such that

- (i)  $a_n < 0$  for  $n = 0, 1, 2, \dots$
- (ii)  $a_1 = -1$  and  $f(r_n) = e^n$  for  $n = 0, 1, 2, \dots$

and put  $g(z) = -f(z)$ . Then clearly  $g(z)$  satisfies all the hypothesis of Theorem 2 and thus,  $g(z) = e^z$  for all  $z$ . Consequently,  $f(z) = -e^z$  for all  $z$ .

This shows that the condition  $a_n > 0$  for  $n = 0, 1, 2, \dots$  cannot be relaxed to the condition  $a_n \neq 0$  for  $n = 0, 1, 2, \dots$ .

### References

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