ENTIRE FUNCTIONS OF ORDER ONE AND INFINITE TYPE

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Abstract. In this paper we first prove an auxiliary result that an entire function of order one and infinite type must have infinitely many non–zero zeros. We then give an explicit canonical representation for those functions. We apply the representation to prove a result and its converse about entire functions of order one and infinite type. Next, we mention a few interesting examples of entire functions of order one and infinite type. Finally, we formulate and disprove a conjecture which serves as an analogue to the Paley–Wiener Theorem for entire functions of order one and infinite type.

In the famous Hadamard Factorization Theorem the canonical product has to be interpreted as the empty product when considering for example functions like $f(z) = e^z$ or more generally $f(z) = e^{g(z)}$ where $g(z)$ is an entire function since these functions have no zeros. If one restricts $f(z)$, for instance, to entire functions of order one and infinite type (or of order one and zero type) the canonical product would then be nonempty.

Lemma. Every entire function $f(z)$ of order one and infinite type must have infinitely many zeros.

Proof. We shall prove this lemma without using the Hadamard Factorization Theorem. If $f(z)$ has no zeros, then let $g(z)$ be a function in the infinite family of entire functions that $\log f(z)$ represents. Then for any integer k, $\log f(z) = g(z) + 2k\pi i$ and thus, $f(z) = e^{g(z)+2k\pi i} = e^{g(z)}$. Using the maximum principle and the definition of order, a simple calculation leads to

$$M_g(r) \leq r$$

for all sufficiently large values of r, where $M_g(r)$ is the maximum modulus function of $g(z)$.

By [1], we see that $g(z)$ is a polynomial $P(z)$ of degree ≤ 1.

If the degree of $P(z)$ were 0 (i.e. $P(z) = B$, a complex number), then $f(z)$ would be of order 0. This contradicts the hypothesis that $f(z)$ is of order 1.

If the degree of $P(z)$ were 1 (i.e. $P(z) = Dz + E$, where D and E are complex numbers with $D \neq 0$), then $f(z)$ would be of order 1 and normal type $|D|$ [2]. This contradicts the hypothesis that $f(z)$ is of infinite type. So $f(z)$ must have at least one zero.

Next we show that $f(z)$ has at least one non–zero zeros. Suppose that $f(z)$ has no non–zero zeros. Then $f(z)$ must have a zero at $z = 0$ (with multiplicity m). Now,

$$f(z) = z^m e^{P(z)}$$

where $P(z)$ is a polynomial whose degree does not exceed 1. Arguing as above we see that the assumption $f(z)$ has no non–zero zeros leads to a contradiction. So $f(z)$ must have at least one non–zero zero.

Finally, we show that $f(z)$ must have infinitely many non–zero zeros. If $f(z)$ has only finitely many non–zero zeros say, $z_1, z_2, \ldots, z_k$, then

$$f(z) = z^m e^{P(z)} \prod_{n=1}^{k} (z - z_n).$$

This would contradict the fact that $f(z)$ is of order 1 and infinite type. The proof of the lemma is now complete.

Combining the lemma with the Hadamard Factorization Theorem we easily get the following.

Theorem 1. Every entire function $f(z)$ of order one and infinite type (which guarantees the existence of infinitely many non–zero zeros) can be represented as

$$f(z) = z^m e^A e^{BZ} Q(z),$$

where

$$Q(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right),$$

m is the multiplicity of the zero of $f(z)$ at $z = 0$ (m could be 0), A and B are complex constants, and $\{z_n\}$ is the set of non–zero zeros.

Remark. It should be emphasized that the above result is different from the Hadamard Factorization Theorem because, unlike the Hadamard Factorization Theorem, it includes representations for such functions as $e^{g(z)}$ where $g(z)$ is an entire function.

The following result also holds.

Theorem 2. Let $f(z)$ be an entire function of order one and infinite type. Let

$$f(z) = z^m e^A e^{BZ} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right).$$

Then,

$$Q(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right)$$

is an entire function of order $\varrho' = 1$ and infinite type.

Proof. If $\varrho' > 1$, then (since the order of $z^m e^A e^{BZ}$ is 0 or 1, 0 when $B = 0$ and 1 when $B \neq 0$) the order ϱ of $f(z)$ would be greater than 1, a contradiction. If $\varrho' < 1$, then $f(z)$ would be of order less than 1 in the case of $B = 0$, (a contradiction to the fact that $f(z)$ is of order 1), or $f(z)$ would be of normal type $|B|$ in the case of $B \neq 0$ (a contradition to the fact that $f(z)$ is of infinite type). Thus, $\varrho' = 1$. It remains to show that $Q(z)$ is of infinite type. We consider two exclusive cases.

Case 1. $B = 0$. If $Q(z)$ were of normal type or type 0, $f(z)$ would be of normal type or of type 0, respectively. This contradicts the fact that $f(z)$ is of infinite type. Thus, in this case, $Q(z)$ is of infinite type.

Case 2. $B \neq 0$. If $Q(z)$ were of type 0, $f(z)$ would be of normal type $|B|$ which is a contradiction. Now, since

$$z^m Q(z) = \frac{f(z)}{e^A e^{Bz}},$$

it follows from Levin [2] that $z^m Q(z)$ is of order one and infinite type. If $Q(z)$ were of normal type, then $z^m Q(z)$ would be of normal type, a contradiction. Also, in this case, $Q(z)$ is of infinite type. Consequently, $Q(z)$ is of order one and infinite type and the proof of Theorem 2 is complete.

Theorem 2 has a converse whose proof is an immediate application of Levin [2] and which we list as

Theorem 3. If $f(z)$ is a function such that

$$f(z) = z^m e^A e^{Bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right),$$

where

$$\prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) \exp(\frac{z}{z_n})$$

is an entire function of order one and infinite type, then $f(z)$ is an entire function of order one and infinite type.

Here are some interesting examples of entire functions of order one and infinite type.

Example 1. The reciprocal of the Gamma function, $\frac{1}{\Gamma(z)}$ is an entire function of order 1 [8] and of infinite type [5].

It is not hard to see that

$$\int_1^{\infty} \left[\frac{1}{\Gamma(x)}\right]^2 < \infty$$

while

$$\int_{-\infty}^{0} \left[\frac{1}{\Gamma(x)}\right]^2 dx = \infty.$$

Consequently,

$$\int_{-\infty}^{\infty} [\frac{1}{\Gamma(x)}]^2 = \infty.$$

This observation will be used later in the paper.

Example 2. For v a non–negative integer, consider the v-th Bessel function

$$J_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{v+2n}}{n\,!\,\Gamma(v+n+1)}.$$

Then, $z^{-v}J_v(z)$ is an entire function of order one [8]. A simple calculation shows that $z^{-v}J_v(z)$ is of infinite type. Now using [2] we see that $J_v(z) = z^v\{z^{-v}J_v(z)\}$ is an entire function of order one and infinite type.

Example 3. $\xi(z) = z(z-1)\pi^{-\frac{z}{2}}\zeta(z)\Gamma(\frac{z}{2})$ is an entire function [7]. Moreover, $\xi(z)$ is of order 1 [7]. Furthermore, if $M(r) = \max\{|\xi(z)| : |z| = r\}$, then [6] $\log M(r) \sim \frac{1}{2} r \log r$ as $r \to \infty$. So,

$$\frac{\log M(r)}{\log r} \sim \frac{1}{2} r$$

as $r \to \infty$ which implies that $\xi(z)$ is of infinite type.

Example 4. The entire function

$$f(z) = \sum_{n=2}^{\infty} \left(\frac{\log n}{n}\right)^n z^n$$

mentioned in [2] is easily verified to be of order one and infinite type.

Now let us consider the following famous theorem.

Theorem (Paley–Wiener). [4] $f(z)$ is an entire function such that

$$|f(z)| \le Ce^{A|z|}$$

for positive constants A and C and all values of z and

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$$

if and only if there exists a function ϕ in $L^2(-A, A)$ such that

$$f(z) = \int_{-A}^{A} \phi(t) e^{izt} dt.$$

We observe that if $f(z)$ is an entire function of exponential type, then $f(z)$ is of order ≤ 1. Moreover, if $f(z)$ is of order < 1, then [1] $f(z)$ is of exponential type 0.

If we combine this with $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$ we see [4] that $f(z) \equiv 0$ which is a trivial case. The more interesting case is when the order of $f(z)$ is 1.

The Paley–Wiener Theorem was generalized by Dzhrbashyan as follows.

Theorem (Dzhrbashyan). [3] The class of entire functions $f(z)$ of order one and type $\leq A$ for which

$$\int_{-\infty}^{\infty} |f(x)|^2 |x|^w dx < \infty \quad (-1 < w < 1)$$

coincides with the set of functions of the form

$$f(z) = \int_{-A}^{A} E_1\{itz; \mu\} Q(t) |t|^{\mu - 1} dt,$$

where $\mu = 1 + \frac{w}{2}$,

$$E_\varrho(u; \mu) = \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(\mu + \frac{k}{\varrho})} (\mu > 0, \varrho > 0)$$

(functions of the Mittag–Leffler type), and $Q(t) \in L^2(-A, A)$.

To see that the Dzhbashyan Theorem generalizes the Paley–Wiener Theorem take $w = 0$. Then $\mu = 1$ and

$$E_1(itz; \mu) = \sum_{k=0}^{\infty} \frac{(itz)^k}{\Gamma(1 + k)} = e^{itz}.$$

The Dzhrbashyan Theorem "characterizes" entire functions of order one and finite type but leaves open the case of entire functions of order one and infinite type. The question now is whether there is an integral representation "analogue" to the Dzhrbashyan Theorem (or the Paley–Wiener Theorem) for entire functions of order one and infinite type.

This question leads to the following conjecture.

Conjecture. $f(z)$ is an entire function of order 1 and infinite type such that $\int_{-\infty}^{\infty}[f(x)]^2 dx = \infty$ if and only if there is ϕ in $L^2(-\infty, \infty)$ such that

$$f(z) = \int_{-\infty}^{\infty} \phi(t) e^{izt} dt.$$

The necessity of the conjecture is false. Consider for instance $f(z) = \frac{1}{\Gamma(z)}$. Example 1 shows that $\frac{1}{\Gamma(z)}$ satisfies the hypotheses. However, the conclusion does not follow because $\lim_{x \to -\infty} \frac{1}{\Gamma(x)} \neq 0$.

It is worth mentioning that the sufficiency of the conjecture is also false. Consider for instance $\phi(t) = 1$ if $t \in [-1, 1]$ and $\phi(t) = 0$, otherwise.

Then $f(z) = \frac{2 \sin z}{z}$ which is of type 1. Alternatively, one can see that

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x} \right)^2 dx = \pi.$$

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