Exploring new identities with Maple as a tool

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Abstract:- Algorithms, like LLL, and Computer Algebra Systems, like Maple, are modern tools that can be used to discover identities. Hopefully, these discoveries can then be coupled with mathematical proofs to become valid. As a result, this provides a unique and excellent venue to advance our knowledge.

Keywords: - Algorithms, Maple, LLL, Identities, CAS, Euler, Zeta

1 Introduction

Let $z = x+iy$ be a complex number. The **Riemann zeta function**, denoted by $\zeta(z)$, is defined by

$$\zeta(z) = \sum_{1}^{\infty} \frac{1}{n^z},$$

with $x > 1$. Euler proved that

$$\frac{\zeta(2n)}{\pi^{2n}}$$

is always a rational number for every whole number n. However, he was unable to determine whether

$\frac{\zeta(3)}{\pi^3}$ is rational or not. This problem has remained unsolved for over 300 years. Using the LLL (Lenstra, Lenstra, Lovasz [2]) algorithm and Maple, we tackle the above problem but remain realistic and hope to gain new insights. In addition, Euler [6] represented $\zeta(2)$ as

$$\zeta(2) = 3\sum_{1}^{\infty} \frac{1}{n^2\binom{2n}{n}}.$$

In this article, we find a similar new representation for $\zeta(3)$ in terms of

$$\sum_{1}^{\infty} \frac{1}{n^3\binom{2n}{n}}$$

using a previous paper of the author [1] and the Computer Algebra System Maple. The importance of the result is also due to the fact that

$$\sum_{1}^{\infty} \frac{1}{n^3\binom{2n}{n}}$$

converges more rapidly than $\sum_{1}^{\infty} \frac{1}{n^3} = \zeta(3)$ which is still an unknown constant.

2 Results

Theorem

$$a)\ \int_0^1 \frac{\log x}{x^3+1}\,dx = \sum_{0}^{\infty} \frac{(-1)^{n+1}}{(3n+1)^2}.$$

$$b)\ \int_0^1 \frac{\log x}{x^3+1}\,dx = -\frac{1}{27}\pi^2 - \frac{1}{2}\left(1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - \cdots\right)$$

$$c)\ 1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - \cdots = \frac{-2\pi^2}{9} + 3\sum_{1}^{\infty} \frac{1}{(3n-2)^2}.$$

Proof. a) Let $x=e^{-t}$. Then expand the denominator as an infinite series and integrate term by term.

b) Splitting n into even and odd, say $n=2k$ or $n=2k+1$ we get

$$\sum_0^\infty \frac{(-1)^{n+1}}{(3n+1)^2} = -\sum_0^\infty \frac{1}{(6k+1)^2} + \sum_0^\infty \frac{1}{(6k+4)^2}.$$

Combining (1) with part (a) we get

$$\int_0^1 \frac{\log x}{x^3+1}dx = -\sum_0^\infty \frac{1}{(6k+1)^2} + \sum_0^\infty \frac{1}{(6k+4)^2}.$$

Moreover,

$$1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\cdots=$$

$$\sum_0^\infty \frac{1}{(6k+1)^2}+\sum_0^\infty \frac{1}{(6k+2)^2}-\sum_0^\infty \frac{1}{(6k+4)^2}-\sum_0^\infty \frac{1}{(6k+5)^2}.$$

Now

$$\frac{\pi^2}{6}=\sum_1^\infty \frac{1}{m^2}$$

$$=\sum_1^\infty \frac{1}{(6k)^2}+\sum_0^\infty \frac{1}{(6k+1)^2}$$

$$+\sum_0^\infty \frac{1}{(6k+2)^2}+\sum_0^\infty \frac{1}{(6k+3)^2}+\sum_0^\infty \frac{1}{(6k+4)^2}+\sum_0^\infty \frac{1}{(6k+5)^2},$$

or

$$\sum_0^\infty \frac{1}{(6k+1)^2}+\sum_0^\infty \frac{1}{(6k+2)^2}+\sum_0^\infty \frac{1}{(6k+3)^2}$$

$$+\sum_0^\infty \frac{1}{(6k+4)^2}+\sum_0^\infty \frac{1}{(6k+5)^2}=\frac{35\pi^2}{432}.$$

Combining this with (3), we can write

$$\sum_0^\infty \frac{1}{(6k+4)^2}+\sum_0^\infty \frac{1}{(6k+5)^2}=\frac{35\pi^2}{432}-\frac{1}{2}\left[1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\cdots\right]$$

$$-\frac{1}{18}\sum_0^\infty \frac{1}{(2k+1)^2}.$$

But $\sum_0^\infty \frac{1}{(2k+1)^2}=\frac{\pi^2}{8}$. Thus

$$\sum_0^\infty \frac{1}{(6k+4)^2}+\sum_0^\infty \frac{1}{(6k+5)^2}=\frac{2\pi^2}{27}-\frac{1}{2}\left[1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\cdots\right].$$

By (2) we can now write this as

$$\int_0^1 \frac{\log x}{x^3+1}dx+\sum_0^\infty \frac{1}{(6k+1)^2}+\sum_0^\infty \frac{1}{(6k+5)^2}=\frac{2\pi^2}{27}-\frac{1}{2}\left[1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\cdots\right].$$

But

$$\sum_0^\infty \frac{1}{(6k+1)^5}+\sum_0^\infty \frac{1}{(6k+2)^5}=1+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{17^2}+\cdots=$$

$$\sum_1^\infty \frac{1}{m^2}-\sum_1^\infty \frac{1}{(2m)^2}-\frac{1}{3^2}\sum_1^\infty \frac{1}{(2k+1)^2}=\frac{\pi^2}{9}.$$

Therefore,

$$\int_0^1 \frac{\log x}{x^3+1}dx+\frac{\pi^2}{9}=-\frac{1}{27}\pi^2-\frac{1}{2}\left(1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\cdots\right)$$

c)

$$1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\frac{1}{10^2}-\frac{1}{11^2}+\cdots=$$

$$\left(1-\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}-\frac{1}{10^2}+\cdots\right)+2\left(\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{8^2}+\frac{1}{10^2}+\cdots\right)=$$

$$\left(1-\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}-\frac{1}{10^2}+\cdots\right)+\frac{1}{2}\left(1-\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}-\frac{1}{10^2}+\cdots\right)=$$

$$=\frac{3}{2}\left(1-\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}-\frac{1}{10^2}+\cdots\right)$$

$$=\frac{3}{2}\left(\sum_0^\infty \frac{1}{(3k+1)^2}-\sum_0^\infty \frac{1}{(3k+2)^2}\right)$$

$$=\frac{3}{2}[2\sum_0^\infty \frac{1}{(3k+1)^2}-\frac{8}{9}\frac{\pi^2}{6}]$$

$$=3\sum_1^\infty \frac{1}{(3k-2)^2}-\frac{2\pi^2}{9}$$

and the proof of the theorem is complete.

In [1] the author found the following representation:

$$\zeta(3)=\frac{\pi}{2}\sum_{1}^{\infty}\frac{\sin(\frac{n\pi}{3})}{n^2}-\frac{3}{4}\sum_{1}^{\infty}\frac{1}{n^3\binom{2n}{n}}$$

Now,

$$\sum_{1}^{\infty}\frac{\sin\left(\frac{n\pi}{3}\right)}{n^2}=\frac{\sqrt{3}}{2}\left(1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\cdots\right)$$

and therefore by (c) of the previous theorem:

$$\sum_{1}^{\infty}\frac{\sin(\frac{n\pi}{3})}{n^2}=\frac{\sqrt{3}}{2}\left(-\frac{2\pi^2}{9}+3\sum_{1}^{\infty}\frac{1}{(3n-2)^2}\right)$$

Thus,

$$\zeta(3)=-\frac{\sqrt{3}}{18}\pi^3+\frac{3\sqrt{3}}{4}\pi\sum_{1}^{\infty}\frac{1}{(3n-2)^2}-\frac{3}{4}\sum_{1}^{\infty}\frac{1}{n^3\binom{2n}{n}}$$

At this point it is natural to hope to express

$$\sum_{1}^{\infty}\frac{1}{(3n-2)^2}$$

in terms of π^2 only, given the shift by 2 of an

already known series, which would give an amazingly pretty result. However, the exact value of the above simply looking series is a difficult open problem and the series $\sum_{0}^{\infty}\frac{1}{(n+\frac{1}{3})^2}$ is known

also as $\psi^{(1)}(\frac{1}{3})$. As a result, we turned to Maple

hoping to discover a numerical relation (which does not, in general of course, constitute a proof since the computer operates on rational approximations of numbers). To do so, we used "Integer relation algorithms" which are main tools for computer-assisted mathematics:

Definitions. Let $r\in\Re^n$ be a given vector. We say that the vector $c\in Z^n$ is an **integer relation** for r if

$$\sum_{1}^{n}c_k r_k=0$$

with at least one non-zero c_k. An **integer relation algorithm** searches therefore for such a non-zero vector c.

Here we use the available LLL (Lenstra, Lenstra, Lovasz [2]) algorithm which is implemented in Maple and known there as the lattice-based relations algorithm. For this to be successful we must have some idea of the result sought. Let us say that we want the value (identity) of:

$$\sum_{-\infty}^{\infty}\frac{1}{(n+\frac{1}{3})^2}.$$

We might expect a multiple of π^2. So we use Maple (the command **evalf** means evaluate using floating-point arithmetic, **trunc** stands for truncate):

>digits:=20;
a:=evalf(sum(1/(n+1/3)∧2,n=0..infinity));
b:=evalf(sum(1/(n+1/3)∧2,n=-infinity..-1));
c:=evalf(Pi∧2);

Digits := 20 a := 10.095597125427094082, b := 3.0638754093587174100, c := 9.8696044010893586191

>A:= trunc(10∧10 *a); B:= trunc(10∧10 *b); C:= trunc(10∧10 *c);

A := 100955971254, B := 30638754093, C := 98696044010

>v1:= [A,1,0,0]; v2:= [B,0,1,0]; v3:= [C,0,0,1]; v1 := [100955971254, 1, 0, 0]; v2 := [30638754093, 0, 1, 0]; v3 := [98696044010, 0, 0, 1]

>readlib(lattice): >lattice([v1,v2,v3], integer);

[[-1, -3, -3, 4],[-137597, 9200, 17061, -14707],[59610, -90913, 137944, 50172]]

This outcome can be tested again with Maple:

```
>evalf(-3* sum(1/(n+1/3)∧2,n=0..infinity)-3* sum(1/(n+1/3)∧2, n=-infinity..-1)+4*Pi∧2 );
```

0

Note that the point of the LLL algorithm is to find vectors whose components are small. As a result, we can write

$$-3\sum_{0}^{\infty}\frac{1}{(n+\frac{1}{3})^2}-3\sum_{-\infty}^{-1}\frac{1}{(n+\frac{1}{3})^2}+4\pi^2=0.$$

That is

$$\sum_{-\infty}^{\infty}\frac{1}{(n+\frac{1}{3})^2}=\frac{4\pi^2}{3}.$$

We can try now to validate the discovered result with a proof. Actually, we shall give four different proofs each being unique in its ideas and is from a different area of mathematics:

Proof 1 (Residue Theory proof). Let

$f(z)=\frac{\pi\cos\pi z}{\sin\pi z}\frac{1}{(3z-2)^2}$. Then f has simple poles at $z=n\in Z$ and a pole of order 2 at $z=\frac{2}{3}$. Now it is easy to see that

$$Res[f(z),z=n]=\frac{1}{(3n-2)^2}.$$

Using Cauchy's Formula, we have

$$Res[f(z),\frac{2}{3}]=-\frac{4\pi^2}{27}.$$

Let S_n denote the square with vertices at

$$\pm(n+\frac{1}{2})\pm(n+\frac{1}{2})i.$$

Then by Cauchy residue theorem we have

$$\int_{S_n}f(z)\,dz=2\pi i\left[-\frac{4\pi^2}{27}+\sum_{k=-n}^{n}Res(f(z),k)\right]=2\pi i\left[-\frac{4\pi^2}{27}+\sum_{-n}^{n}\frac{1}{(3k-2)^2}\right].$$

To prove that $\int_{S_n}f(z)\,dz\to 0$ as $n\to\infty$, we first show that $|\frac{\cos\pi z}{\sin\pi z}|\le M$, where

$$M=\max(1,\frac{1+e^{-\pi}}{1-e^{-\pi}}),\forall z\in S_n.$$

On the horizontal line segment,

$$z=-i(n+\frac{1}{2})+x \text{ with } |x|\le n+\frac{1}{2},$$

$|\cot\pi z|=\frac{|1+e^{2\pi iz}|}{|1-e^{2\pi iz}|}$. But $|e^{2\pi iz}|=e^{-2\pi y}$. So

$$|\cot\pi z|\le\frac{1+e^{-2\pi(n+\frac{1}{2})}}{1+e^{-2\pi(n+\frac{1}{2})}}=\frac{1+e^{-\pi(2n+1)}}{1+e^{-\pi(2n+1)}}\le\frac{1+e^{-\pi}}{1-e^{-\pi}}.$$

Similarly for the line segment

$$z=-i(n+\frac{1}{2})+x \text{ with } |x|\le n+\frac{1}{2}.$$

On the vertical line segment,

$$z=iy+(n+\frac{1}{2}) \text{ with } |y|\le n+\frac{1}{2},$$

$$|\cot\pi z|=|\cot(\pi iy+(n+\frac{1}{2})\pi|=|\tan(\pi iy)|$$

$$=|\tan(\pi y)|=\frac{e^{\pi y}-e^{-\pi y}}{e^{\pi y}+e^{-\pi y}}\le 1.$$

Similarly for the line segment

$$z=iy-(n+\frac{1}{2})\ with\,|y|\le n+\frac{1}{2}.$$

So

$$|\int_{S_n}f(z)dz\le\sup|\frac{\pi\cos\pi z}{\sin\pi z}\frac{1}{(3z-2)^2}||\int_{S_n}dz|$$

$$\le M\sup|\frac{1}{(3z-2)^2}|.4.(2n+1)$$

$$\leq 4M\frac{2n+1}{(3n-2)^2}.$$

Therefore

$$\int_{S_n} f(z)dz \to 0 \text{ as } n\to\infty.$$

Thus

$$-\frac{4\pi^2}{27}+\sum_{-n}^{n}\frac{1}{(3k-2)^2}\to 0$$

as $n\to\infty$.

Consequently,

$$\sum_{-\infty}^{\infty}\frac{1}{(3n-2)^2}=\frac{4\pi^2}{27},$$

or

$$\sum_{-\infty}^{\infty}\frac{1}{(3n+1)^2}=\frac{4\pi^2}{27},$$

which agrees with the result we discovered by Maple.

Proof 2 (Representation proof). A positive integer n can be represented as $3k$, $3k$-1, or $3k$-2 where k is a positive integer. Thus

$$\sum_{1}^{\infty}\frac{1}{n^2}=\sum_{1}^{\infty}\frac{1}{(3k)^2}+\sum_{1}^{\infty}\frac{1}{(3k-1)^2}+\sum_{1}^{\infty}\frac{1}{(3k-2)^2}$$

and therefore

$$\sum_{1}^{\infty}\frac{1}{(3k-1)^2}+\sum_{1}^{\infty}\frac{1}{(3k-2)^2}=\frac{4\pi^2}{27}.$$

In the second series, let k=m+1. Then

$$\sum_{1}^{\infty}\frac{1}{(3k-1)^2}+\sum_{0}^{\infty}\frac{1}{(3m+2)^2}=\frac{4\pi^2}{27}.$$

We now write this as

$$\sum_{1}^{\infty}\frac{1}{(3k-1)^2}+\sum_{0}^{\infty}\frac{1}{(-3m-2)^2}=\frac{4\pi^2}{27}.$$

Finally, let t=-m. So

$$\sum_{1}^{\infty}\frac{1}{(3k-1)^2}+\sum_{-\infty}^{0}\frac{1}{(3t-2)^2}=\frac{4\pi^2}{27}$$

and consequently

$$\sum_{-\infty}^{\infty}\frac{1}{(3n+1)^2}=\frac{4\pi^2}{27}.$$

Proof 3 (Most simple). Explicitly

$$\sum_{-\infty}^{\infty}\frac{1}{(3n+1)^2}=\cdots+\frac{1}{8^2}+\frac{1}{5^2}+\frac{1}{2^2}+1+\frac{1}{4^2}+\frac{1}{7^2}+\cdots$$

Observe that the above terms are terms of the series

$\sum_{1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ which has in addition the terms

$$\frac{1}{3^2}+\frac{1}{6^2}+\frac{1}{9^2}+\cdots=\sum_{1}^{\infty}\frac{1}{(3n)^2}=\frac{\pi^2}{54}.$$

Therefore

$$\sum_{-\infty}^{\infty}\frac{1}{(3n+1)^2}=\frac{\pi^2}{6}-\frac{\pi^2}{54}=\frac{4\pi^2}{27}.$$

The next proof is very general which, in addition, leads to a quite simple proof of an extremely important corollary giving Euler's explicit formulas for $\zeta(2k)$ as we shall see.

Proof 4 (Most general). The series

$$\sum_{-\infty}^{\infty}\frac{1}{(n-z)^2}$$

converges uniformly on each compact subset of the

plane that avoids the integers. To see this, let A be such set which we can always enclose in an open disk centered at (0,0) of some radius R. Now for each point z of A, we have $|n-z|\geq n-|z|>n-R$ and for each integer $t>R$ we then have for integers n with $|n|\geq$ t:

$$\sum_{-\infty}^{\infty}\frac{1}{|n-z|^2}\leq 2\sum_{n=t}^{\infty}\frac{1}{(n-R)^2}$$

which is the remainder of an absolutely convergent series and converges to 0 as t tends to infinity and since this does not depend on z, the convergence is uniform on A. Next, since the series

$$\sum_{-\infty}^{\infty}\frac{1}{(n-z)^2}$$

converges absolutely on its domain of definition, we can replace n by n+1 below to get

$$\sum_{-\infty}^{\infty}\frac{1}{(n-(z+1))^2}=\sum_{-\infty}^{\infty}\frac{1}{((n+1-(z+1))^2}=\sum_{-\infty}^{\infty}\frac{1}{(n-z)^2}.$$

Now let b>0 and let z=x+iy with $|y|\geq b$. Let k be the integral part of x. Then

$$|\sum_{-\infty}^{\infty}\frac{1}{(n-(x+iy))^2}|=|\sum_{-\infty}^{\infty}\frac{1}{(n-(x-p+iy))^2}|$$

$$=\lim_{d\to\infty,\ t\to\infty}|\sum_{t}^{d}\frac{1}{(n-(x-p+iy))^2}|$$

$$\leq\lim_{d\to\infty,\ t\to\infty}\sum_{t}^{d}\frac{1}{|(n-(x-p+iy))|^2}$$

$$=\sum_{-\infty}^{\infty}\frac{1}{|n-x+p-iy|^2}$$

$$=\sum_{1}^{\infty}\frac{1}{|n-x+p-iy|^2}+\sum_{0}^{\infty}\frac{1}{|-n-x+p-iy|^2}$$

$$\leq\sum_{1}^{\infty}\frac{1}{(n-1)^2+b^2}+\sum_{0}^{\infty}\frac{1}{n^2+b^2}=2\sum_{0}^{\infty}\frac{1}{n^2+b^2}.$$

Next the function

$$g(z)=\sum_{-\infty}^{\infty}\frac{1}{(n-z)^2}-\frac{\pi^2}{\sin^2\pi z}$$

is holomorphic except on the integers and satisfies

$$\sum_{-\infty}^{\infty}\frac{1}{(n-(z+1))^2}-\frac{\pi^2}{\sin^2\pi(z+1)}=\sum_{-\infty}^{\infty}\frac{1}{(n-z)^2}-\frac{\pi^2}{\sin^2\pi z}$$

and

$$|\sum_{-\infty}^{\infty}\frac{1}{(n-z)^2}-\frac{\pi^2}{\sin^2\pi z}|\leq\sum_{0}^{\infty}\frac{1}{b^2+n^2}+\frac{2\pi^2}{(e^{\pi b}-e^{-\pi b})^2}$$

which tends to 0 when b tends to infinity. So the function

$$g(z)=\sum_{-\infty}^{\infty}\frac{1}{(n-z)^2}-\frac{\pi^2}{\sin^2\pi z}$$

is bounded on $[0,1]+i\,\Re$ and thus on C. Thus the function g is constant; since $g(ib)$ tends to 0 as b tends to infinity, g must be the zero function. Thus

$$\sum_{-\infty}^{\infty}\frac{1}{(n-z)^2}=\frac{\pi^2}{\sin^2\pi z}$$

from which we get (take the special case z= 2/3)

$$\sum_{-\infty}^{\infty}\frac{1}{(3n-2)^2}=\frac{4\pi^2}{27}.$$

That is,

$$\sum_{-\infty}^{\infty}\frac{1}{(3n+1)^2}=\frac{4\pi^2}{27}.$$

Corollary. $\sum_{k=1}^{\infty}\frac{1}{k^{2n}}=\frac{(-1)^{n-1}2^{2n-1}B_{2n}\pi^{2n}}{(2n)!}$, where B_n denote the Bernoulli numbers defined by

$$\frac{z}{e^z-1}=\sum_{0}^{\infty}B_n\frac{z^n}{n!}.$$

Proof. The function

$$\frac{\pi^2}{\sin^2 \pi z} - \sum_{-\infty}^{\infty} \frac{1}{(n-z)^2}$$

is the derivative of

$$\frac{1}{z} + \sum_{1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n} \right) - \pi \cot \pi z$$

and this derivative is zero. This implies that

$\pi \cot \pi z - \frac{1}{z} - \sum_{1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n} \right)$ is constant and, moreover, it turns out that this constant is 0. Therefore,

$$\pi z \cot \pi z = 1 + \sum_{1}^{\infty} \frac{2z^2}{z^2 - k^2}.$$

Let a_n be the coefficient of z^n in the power series expansion of the function $\pi z \cot \pi z$. Since

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{dz}{z^{n+1}} + \frac{1}{2\pi i} \sum_{k=1}^{\infty} \int_{|z|=r} \frac{2z^{1-n} dz}{z^2 - k^2}$$

for every n=0,1,2,…, $a_0 = 1$ *and* $a_n = 0$ *for odd* n. Now for even n we have

$$a_n = 2 \sum_{k=1}^{\infty} \frac{1}{k^n}.$$

Comparing with the power series expansion

$$\pi z \cot \pi z = 1 + \sum_{1}^{\infty} \frac{(-4)^n B_{2n} \pi^{2n} z^{2n}}{(2n)!},$$

we get

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n-1} B_{2n} 2^{2n-1} \pi^{2n}}{(2n)!}.$$

3 Concluding comments

Even though $\zeta(6) = \frac{\pi^6}{945}$, extensive computation [3] has ruled out finding moderately sized integers a and b such that

$$\zeta(6) = \frac{a}{b} \sum_{1}^{\infty} \frac{1}{n^6 \binom{2n}{n}}.$$

In 1979, Apéry proved that $\zeta(3)$ is irrational but it remains open whether $\frac{\zeta(3)}{\pi^3}$ is irrational (the feeling is that it is because extensive computation has been done to suggest that if $\zeta(3) = \frac{a}{b}\pi^3$ for some integers a and b, then a and b are astronomically large). It would therefore be very interesting, though very difficult, to look for an identity for $\zeta(3)$ in terms of π^3, possibly searching for an irrational constant c (involving log 2) such that $\zeta(3) = c\pi^3$ since Euler himself has conjectured (see, for instance, [4, p. 1078] or [5, p. 149]) that for odd n, $\zeta(n) = f(n, \log 2)\pi^n$.

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