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Generalized Integration Formulas

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Abstract

The purpose of this paper is three-fold. First, we generalize formula in [1]. Next, we consider an open problem. Finally, we supply a proof of formula 3 in section 3.415 of [2].

1 Generalizing an integral formula

Formula 135 on page 171 in [1], states that

$$\int_0^\infty \frac{\arctan x}{e^{2\pi x}+1}dx = \frac{3}{4}\log 2 - \frac{1}{2}.$$

In this section we generalize this formula for a parameter p (instead of just 2π) as

$$\int_0^\infty \frac{\arctan x}{e^{px}+1}dx = \frac{1}{2}\log\frac{p}{2\pi} - \frac{1}{2} + \frac{\pi}{2p}\log(2\pi) - \frac{\pi}{p}\log\Gamma(\frac{p}{2\pi}+\frac{1}{2}),$$

where $\Gamma(x)$ denotes the famous Gamma Function.
To prove this we use Binet's Second Formula for the Gamma function (for a proof, see [5], pp. $250-251$):

$$\log\Gamma(z) = (z-\frac{1}{2})\log z - z + \frac{1}{2}\log(2\pi) + 2\int_0^\infty \frac{\arctan(\frac{t}{z})}{e^{2\pi t}-1}dt, Re(z)>0$$

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and the Duplication Formula for the Gamma function (for a proof, see [3], pp. 50 – 51):

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$$

to get

$$\log\Gamma(z+\frac{1}{2}) = z\log z - z + \frac{1}{2}\log(2\pi) - 2\int_0^\infty \frac{\arctan(\frac{t}{z})}{e^{2\pi t}+1}dt.$$

In particular, for $z = \frac{p}{2\pi}$, we have

$$\log\Gamma(\frac{p}{2\pi}+\frac{1}{2}) = \frac{p}{2\pi}\log\frac{p}{2\pi} - \frac{p}{2\pi} + \frac{1}{2}\log(2\pi) - 2\int_0^\infty \frac{\arctan(\frac{2\pi t}{p})}{e^{2\pi t}+1}dt.$$

Now letting $x = \frac{2\pi t}{p}$ yields

$$\log\Gamma(\frac{p}{2\pi}+\frac{1}{2}) = \frac{p}{2\pi}\log\frac{p}{2\pi} - \frac{p}{2\pi} + \frac{1}{2}\log(2\pi) - \frac{p}{\pi}\int_0^\infty \frac{\arctan x}{e^{px}+1}dx$$

or

$$\int_0^\infty \frac{\arctan x}{e^{px}+1}dx = \frac{1}{2}\log\frac{p}{2\pi} - \frac{1}{2} + \frac{\pi}{2p}\log(2\pi) - \frac{\pi}{p}\log\Gamma(\frac{p}{2\pi}+\frac{1}{2}).$$

2 Relation to an open problem

Euler psi function $\psi(x)$ is defined by $\psi(x) = \frac{d\log\Gamma(x)}{dx}$. By ([4], p. 27)

$$\psi^{(n)}(x) = (-1)^{n+1}n!\sum_{k=0}^\infty \frac{1}{(x+k)^{n+1}}.$$

Finding the exact value of $\psi'(\frac{1}{3})$ is an open problem. We express this as an integral related to the above type. First, rewrite $\psi'(\frac{1}{3}) = \sum_0^\infty \frac{1}{(k+\frac{1}{3})^2}$ as

$$\psi'(\frac{1}{3}) = \sum_1^\infty \frac{1}{(k-\frac{2}{3})^2} = 9\sum_1^\infty \frac{1}{(3k-2)^2}.$$

We use Residue Theory where it is natural therefore to consider the function $f(z) = \frac{\pi\cos\pi z}{\sin\pi z}\frac{1}{(3z-2)^2}$. Then f has simple poles at $z = k \in \mathbb{Z}$ and a pole of order 2 at $z = \frac{2}{3}$.

Now it is easy to see that

$$Res[f(z), z = k] = \frac{1}{(3k-2)^2}.$$

Using Cauchy's Formula, we have

$$Res[f(z), \frac{2}{3}] = -\frac{4\pi^2}{27}.$$

Let R_k denote the rectangle with vertices at

$$\frac{1}{2}, \pm(k+\frac{1}{2})i, k+\frac{1}{2} \pm (k+\frac{1}{2})i.$$

Then by Cauchy residue theorem we have

$$\int_{R_k} f(z)dz = 2\pi i\left[-\frac{4\pi^2}{27} + \sum_{n=1}^{k} Res(f(z), n)\right] = 2\pi i\left[-\frac{4\pi^2}{27} + \sum_{1}^{k} \frac{1}{(3n-2)^2}\right].$$

On the horizontal line segment, $z = i(k+\frac{1}{2}) + x$ with $\frac{1}{2} \leq x \leq k + \frac{1}{2}$, $|\cot\pi z| = \frac{|1+e^{2\pi iz}|}{|1-e^{2\pi iz}|}$. But $|e^{2\pi iz}| = e^{-2\pi y}$. So

$|\cot\pi z| \leq \frac{1+e^{-2\pi(k+\frac{1}{2})}}{1+e^{-2\pi(k+\frac{1}{2})}} = \frac{1+e^{-\pi(2k+1)}}{1-e^{-\pi(2k+1)}} \leq \frac{1-e^{-\pi}}{1-e^{-\pi}} = \frac{e^{\pi}+1}{e^{\pi}-1}$.

On the horizontal line segment $z = -i(k+\frac{1}{2}) + x$ with $\frac{1}{2} \leq x \leq k + \frac{1}{2}$, $|\cot\pi z| = \frac{|e^{2\pi iz}+1|}{|e^{2\pi iz}-1|}$. But $|e^{2\pi iz}| = e^{-2\pi y}$. So

$|\cot\pi z| \leq \frac{e^{-2\pi(-k-\frac{1}{2})}+1}{e^{-2\pi(-k-\frac{1}{2})}-1} = \frac{e^{\pi(2k+1)}+1}{e^{\pi(2k+1)}-1} = 1 + \frac{2}{e^{\pi(2k+1)}-1} \leq 1 + \frac{2}{e^{\pi}-1} = \frac{e^{\pi}+1}{e^{\pi}-1}$.

On the vertical line segment, $z = iy + (k+\frac{1}{2})$ with $|y| \leq k + \frac{1}{2}$, $|\cot\pi z| = |\cot(\pi iy + (k+\frac{1}{2})\pi)| = |\tan(\pi iy)| = |\tanh(\pi y)| = \frac{e^{\pi y}-e^{-\pi y}}{e^{\pi y}+e^{-\pi y}} \leq 1$.

Thus on these three line segments, $|\frac{\cos\pi z}{\sin\pi z}| \leq M$, where $M = \max(1, \frac{e^{\pi}+1}{e^{\pi}-1})\forall z$ on the three line segments. So along the broken line B_k resulting from the above three line segments, we have

$$|\int_{B_k} f(z)dz| \leq \sup|\frac{\pi\cos\pi z}{\sin\pi z}\frac{1}{(3z-2)^2}||\int_{B_k} dz| \leq M\sup|\frac{1}{(3z-2)^2}|.(4k+3)$$

$$\leq M\frac{4k+3}{(3k-2)^2}.$$

Therefore $\int\limits_{B_k} f(z)dz \to 0$ as $k \to \infty$.

Now let L_k denote the line segment from $(\frac{1}{2}, k+\frac{1}{2})$ to $(\frac{1}{2}, -k-\frac{1}{2})$. We have

$$\int\limits_{L_k} f(z)dz = -\pi \int_{-k-\frac{1}{2}}^{k+\frac{1}{2}} \frac{\cos(\frac{\pi}{2}+i\pi y)}{\sin(\frac{\pi}{2}+i\pi y)} \frac{1}{(-\frac{1}{2}+3iy)^2} idy$$

$$= -\pi \int_{-k-\frac{1}{2}}^{k+\frac{1}{2}} \frac{\sinh(\pi y)}{\cosh(\pi y)} \frac{1}{(\frac{1}{2}-3iy)^2} dy$$

$$= -4\pi \int_{-k-\frac{1}{2}}^{k+\frac{1}{2}} \tanh(\pi y) \frac{1}{(1-6iy)^2} dy = -4\pi \int_{-k-\frac{1}{2}}^{k+\frac{1}{2}} \tanh(\pi y) \frac{1}{1-12iy-36y^2} dy$$

$$= -4\pi \int_{-k-\frac{1}{2}}^{k+\frac{1}{2}} \tanh(\pi y) \frac{1-36y^2+12iy}{(1+36y^2)^2} dy$$

$$= -4\pi \int_{-k-\frac{1}{2}}^{k+\frac{1}{2}} \frac{\tanh(\pi y)(1-36y^2)}{(1+36y^2)^2} dy - 48\pi i \int_{-k-\frac{1}{2}}^{k+\frac{1}{2}} \frac{y\tanh(\pi y)}{(1+36y^2)^2} dy,$$

where in the first integral the integrand is an odd function on $[-(k+\frac{1}{2}), (k+\frac{1}{2})]$ and in the second integral the integrand is an even function on $[-(k+\frac{1}{2}), (k+\frac{1}{2})]$. As a result we can write

$$\int\limits_{L_k} f(z)dz = -48\pi i \int_0^{k+\frac{1}{2}} \frac{2y\tanh(\pi y)}{(1+36y^2)^2} dy = -\frac{4\pi i}{3} \int_1^{1+36(k+\frac{1}{2})^2} \frac{\tanh(\frac{\pi\sqrt{w-1}}{6})}{w^2} dw$$

Therefore

$$-\frac{1}{3} \int_1^{1+36(k+\frac{1}{2})^2} \frac{\tanh(\frac{\pi\sqrt{w-1}}{6})}{w^2} dw = -\frac{2\pi^2}{27} + \frac{1}{2} \sum_1^k \frac{1}{(3n-2)^2}.$$

Thus

$$-\frac{1}{3} \int_1^{1+36(k+\frac{1}{2})^2} \frac{e^{\frac{\pi\sqrt{w-1}}{3}}-1}{(e^{\frac{\pi\sqrt{w-1}}{3}}+1)w^2} dw = -\frac{2\pi^2}{27} + \frac{1}{2} \sum_1^k \frac{1}{(3n-2)^2}.$$

Hence

$$-\frac{1}{3} \int_1^{1+36(k+\frac{1}{2})^2} \frac{1}{w^2} dw + \frac{2}{3} \int_1^{1+36(k+\frac{1}{2})^2} \frac{1}{(e^{\frac{\pi\sqrt{w-1}}{3}}+1)w^2} dw = -\frac{2\pi^2}{27} + \frac{1}{2} \sum_1^k \frac{1}{(3n-2)^2}.$$

Evaluating the first integral and letting $k \to \infty$ we obtain

$$-\frac{1}{3} + \frac{2}{3}\int_1^\infty \frac{1}{(e^{\frac{\pi\sqrt{w-1}}{3}}+1)w^2}dw = -\frac{2\pi^2}{27} + \frac{1}{2}\sum_1^\infty \frac{1}{(3n-2)^2}.$$

Let $w - 1 = u^2, u \geq 0$. Then

$$-\frac{2\pi^2}{27} + \frac{1}{2}\sum_1^\infty \frac{1}{(3n-2)^2} = -\frac{1}{3} + \frac{4}{3}\int_0^\infty \frac{u}{(e^{\frac{\pi u}{3}}+1)(1+u^2)^2}du.$$

Now the integral can be rewritten as

$$\int_0^\infty (e^{\frac{\pi u}{3}}+1)^{-1} d(-\frac{1}{2(1+u^2)})$$

which on integration by parts yields

$$\frac{1}{4} - \frac{\pi}{6}\int_0^\infty \frac{e^{\frac{\pi u}{3}}}{(e^{\frac{\pi u}{3}}+1)^2(1+u^2)}du.$$

Consequently,

$$\sum_1^\infty \frac{1}{(3n-2)^2} = \frac{4\pi^2}{27} - \frac{4\pi}{9}\int_0^\infty \frac{e^{\frac{\pi u}{3}}}{(e^{\frac{\pi u}{3}}+1)^2(1+u^2)}du,$$

where, in the following section, we'll see the relevance of this integral to that of the first section.

3 A proof of a general formula

At this stage, with p denoting a parameter between 0 and π, our interest shifts to the integral $\int \frac{\arctan x}{1+e^{px}}dx$ because of its relevance to our problem as we shall see next:
Integrating by parts using $t = \arctan x$ and $dv = \frac{dx}{1+e^{px}}$ yields

$$\int \frac{\arctan x}{1+e^{px}}dx = \frac{\arctan x}{p}\log\frac{e^{px}}{1+e^{px}} - \frac{1}{p}\int \frac{1}{1+x^2}\log\frac{e^{px}}{1+e^{px}}dx.$$

Differentiating the last integral with respect to the parameter p, it gives $\int \frac{x}{1+x^2}\frac{1}{1+e^{px}}dx$ which if we differentiate again, with respect to p, we get

$$-\int \frac{x^2 e^{px}dx}{(1+x^2)(1+e^{px})^2} = -\int \frac{(x^2+1-1)e^{px}dx}{(1+x^2)(1+e^{px})^2}$$

$$= -\int \frac{e^{px}dx}{(1+e^{px})^2} + \int \frac{e^{px}dx}{(1+x^2)(1+e^{px})^2} = \frac{1}{p(1+e^{px})} + \int \frac{e^{px}dx}{(1+x^2)(1+e^{px})^2}$$

and the relevance is clear upon taking $p = \frac{\pi}{3}$.
To wrap things up it is necessary to consider $\int \frac{1}{1+z^2}\frac{e^{pz}}{(1+e^{pz})^2}dz$. First, applying integration by parts with $u = (1+z^2)^{-1}$ and $dv = \frac{e^{pz}dz}{(1+e^{pz})^2}$ we get

$$\int_C \frac{1}{1+z^2}\frac{e^{pz}}{(1+e^{pz})^2}dz = -\frac{1}{p(1+e^{pz})(1+z^2)}|_C - \frac{2}{p}\int_C \frac{1}{(1+z^2)^2}\frac{z}{(1+e^{pz})}dz$$

$$= -\frac{2}{p}\int_C \frac{1}{(1+z^2)^2}\frac{z}{(1+e^{pz})}dz$$

since C is closed. Next we use Residue Theory on the last integral:
For $R > 0$, take the contour $C = [-R, R] \cup C_R$, where C_R is the circular arc $z = Re^{i\theta}, 0 < \theta < \pi$.
Inside C there is a simple pole of the function $\frac{1}{(1+z^2)^2}\frac{z}{(1+e^{pz})}$ at $z = (2m+1)\frac{\pi}{p}i, m = 0, 1, 2, ..., [\frac{pR}{2\pi} + \frac{1}{2}]$, with residues $-\frac{(2m+1)\frac{\pi}{p}i}{p[(2m+1)^2\frac{\pi^2}{p^2}-1]^2}$ and a double pole at $z = i$ with residue $\frac{p}{8(1+\cos p)}i$. By the Residue Theorem we have

$$\int_{-R}^{R} \frac{e^{px}}{(1+e^{px})^2(1+x^2)}dx + \int_{C_R} \frac{e^{pz}}{(1+e^{pz})^2(1+z^2)}dz$$

$$= \frac{\pi}{2(1+\cos p)} - \frac{4\pi}{p^2}\sum_{m=0}^{[\frac{pR}{2\pi}+\frac{1}{2}]} \frac{(2m+1)\frac{\pi}{p}}{[(2m+1)^2\frac{\pi^2}{p^2}-1]^2}.$$

Now the integrand in the first integral is an even function and as a result

$$2\int_{0}^{R} \frac{e^{px}}{(1+e^{px})^2(1+x^2)}dx + \int_{C_R} \frac{e^{pz}}{(1+e^{pz})^2(1+z^2)}dz$$

$$= \frac{\pi}{2(1+\cos p)} - \frac{4\pi}{p^2}\sum_{m=0}^{[\frac{pR}{2\pi}+\frac{1}{2}]} \frac{(2m+1)\frac{\pi}{p}}{[(2m+1)^2\frac{\pi^2}{p^2}-1]^2}.$$

Next, for large R,

$$|\int_{C_R} \frac{e^{pz}}{(1+e^{pz})^2(1+z^2)}dz| \le \int_{C_R} \frac{|e^{pz}|}{|1+z^2||1+e^{pz}|^2}|dz| \le \frac{\pi Re^{p\Re z}}{(R^2-1)(e^{p\Re z}-1)} \to 0$$

as $R \to \infty$.
Hence

$$\int_0^\infty \frac{e^{px}}{(1+e^{px})^2(1+x^2)}dx = \frac{\pi}{4(1+\cos p)} - \frac{2\pi^2}{p^3}\sum_{m=0}^{\infty}\frac{2m+1}{[(2m+1)^2\frac{\pi^2}{p^2}-1]^2}$$

$$= \frac{\pi}{4(1+\cos p)} - \frac{2\pi^2}{p^3}\sum_{m=0}^{\infty}\{\frac{\frac{p}{4\pi}}{[(2m+1)\frac{\pi}{p}-1]^2} - \frac{\frac{p}{4\pi}}{[(2m+1)\frac{\pi}{p}+1]^2}\}$$

$$= \frac{\pi}{4(1+\cos p)} - \frac{\pi}{2p^2}\sum_{m=0}^{\infty}\{\frac{1}{[(2m+1)\frac{\pi}{p}-1]^2} - \frac{1}{[(2m+1)\frac{\pi}{p}+1]^2}\}.$$

As an important special case, we get

$$\int_0^\infty \frac{e^{\frac{\pi}{3}x}}{(1+e^{\frac{\pi}{3}x})^2(1+x^2)}dx$$

$$= \frac{\pi}{6} + \frac{9}{8\pi}\{\sum_1^\infty \frac{1}{(3n-1)^2} - \sum_1^\infty \frac{1}{(3n-2)^2}\}.$$

Tracing back, we have

$$\int_0^\infty \frac{1}{1+x^2}\log\frac{e^{px}}{1+e^{px}}dx = -\frac{p}{2}\log\frac{p}{2\pi} + \frac{p}{2} - \frac{\pi}{2}\log(2\pi) + \pi\log\Gamma(\frac{p}{2\pi}+\frac{1}{2}).$$

Differentiating with respect to p and tracing back we see that

$$\int_0^\infty \frac{x}{1+x^2}\frac{1}{1+e^{px}}dx = -\frac{1}{2}\log\frac{p}{2\pi} + \frac{1}{2}\frac{\Gamma'(\frac{p}{2\pi}+\frac{1}{2})}{\Gamma(\frac{p}{2\pi}+\frac{1}{2})}.$$

Differentiating with respect to p and tracing back we see that

$$-\frac{1}{2p} + \int_0^\infty \frac{e^{px}dx}{(1+x^2)(1+e^{px})^2} = -\frac{1}{2p} + \frac{1}{4\pi}\frac{d(\frac{\Gamma'(\frac{p}{2\pi}+\frac{1}{2})}{\Gamma(\frac{p}{2\pi}+\frac{1}{2})})}{d(\frac{p}{2\pi}+\frac{1}{2}))}.$$

Then

$$\int_0^\infty \frac{e^{px}dx}{(1+x^2)(1+e^{px})^2} = \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{1}{(n+\frac{p}{2\pi}+\frac{1}{2})^2}.$$

Application. The table of integrals (given without proofs) in [2] is probably the most widely used one. We supply the proof of a quite general formula (Formula 3 in section 3.415):

$$\int_0^\infty \frac{xdx}{(x^2+\beta^2)(e^{\mu x}+1)} = \frac{1}{2}[\psi(\frac{\beta\mu}{2\pi}+\frac{1}{2}) - \log(\frac{\beta\mu}{2\pi})], Re\beta > 0, Re\mu > 0.$$

First, write the integral as

$$\int_0^\infty \frac{\frac{x}{\beta}}{(1+(\frac{x}{\beta})^2)(e^{\frac{\mu\beta x}{\beta}}+1)} d(\frac{x}{\beta})$$

and use our earlier result to get

$$\int_0^\infty \frac{xdx}{(x^2+\beta^2)(e^{\mu x}+1)} = -\frac{1}{2}\log\frac{\mu\beta}{2\pi}+\frac{1}{2}\frac{\Gamma'(\frac{\mu\beta}{2\pi}+\frac{1}{2})}{\Gamma(\frac{\mu\beta}{2\pi}+\frac{1}{2})} = -\frac{1}{2}\log\frac{\mu\beta}{2\pi}+\frac{1}{2}\psi(\frac{\mu\beta}{2\pi}+\frac{1}{2}).$$

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