JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **199,** 469–477 (1996)
ARTICLE NO. 0154

On Approximation by a Nonfundamental Sequence of Translates

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Communicated by E. W. Cheney

Received January 27, 1994

If $f(t)$ and its Fourier transform $F(t)$ satisfy some growth conditions and if $\{c_n\}_0^\infty$ is a sequence of distinct real numbers satisfying a certain separation condition, we represent those functions $g(t)$ which are in the closure of the linear span of a nonfundamental sequence $\{f(c_n - t)\}$ in $L_2(\mathbf{R})$. A result about the degree of approximation is also proved.

Let Σ denote the sum with index from 0 to ∞, Σ' denote the sum with nonvanishing denominator, and $\prod^{(k)}$ denote the product with the k term deleted.

Given a function $f(t)$, the Fourier transform $F(x)$ is defined as

$$F(x) = (2\pi)^{-1/2} \int_{\mathbf{R}} \exp(xti) f(t)\, dt.$$

A sequence of functions is fundamental in a space X if the linear span of the elements of the sequence is dense in X. Wiener's classical Tauberian Theorem [6] states that if $f(t) \in L_2(\mathbf{R})$ then the linear span of the set $\{f(c-t)\}_{c \in \mathbf{R}}$ is dense in $L_2(\mathbf{R})$ if and only if $F(t) \neq 0$ a.e. The natural problem as to under what conditions the linear span of sequence $\{f(c_n - t)\}$ is dense in $L_2(\mathbf{R})$ has been studied by Zalik [7] and Faxén [5] among others.

Suppose that $f(t)$ is a continuous function in $L_2(\mathbf{R})$. Assuming that $\{c_n\}$ is a sequence of distinct real numbers such that $|c_n^2 - c_r^2| \geq \rho |n - r| (\rho > 0)$, $\Sigma'(1/c_n^2) < \infty$, and $f(t)$ and $F(t)$ are functions satisfying $f(t) = O\{\exp(-\alpha t^2)\}$, $F(t) = O\{\exp(-at^2)\}$ as $t \to \infty$, $\exp(-bt^2)/F(t) \in L_2(\mathbf{R})$ (α, a, and b some positive numbers), Zalik [1] found a representation for those functions $g(t)$ which are in the closure of the linear span of a nonfundamental sequence in $L_2(\mathbf{R})$ of the form $\{f(c_n - t)\}$. It is notewor-

0022-247X/96 $18.00

thy here that $\Sigma'(1/c_n^2) < \infty$ and $\exp(-bt^2)/F(t) \in L_2(\mathbf{R})$ imply the nonfundamentality of the sequence [5, p. 273, Theorem 2].

In this paper we assume that $\{c_n\}$ is a sequence of distinct real numbers satisfying the separation condition

$$\big| |c_n|^p - |c_r|^p \big| \geq \rho |n - r|$$

for some integer $p > 0$ ($\rho > 0$), $\Sigma'(1/|c_n|^p) < \infty$ and $f(t)$ and $F(t)$ are functions satisfying

$$f(t) = O\{\exp(-\alpha t^2)\},$$

$$F(t) = O\{\exp(-at^2)\} \text{ as } |t| \to \infty, \ \frac{\exp(-bt^2)}{F(t)} \in L_2(\mathbf{R}).$$

Under the previous conditions we obtain the following result:

LEMMA. *For every μ such that $0 < \mu < 1/2b$, there are continuous functions $l_k(\mu, t)$ having Fourier transforms $m_k(\mu, t)$, such that*

(a) *If $h(t) = \exp(-bt^2)/|F(t)|$, then*

$$|m_k(\mu, t)| \leq d \exp\left\{-\left(\frac{1}{2\mu} - b\right)t^2 + \mu\left(\frac{c_k^2 + |c_k|^p}{2}\right)\right\} h(t),$$

where d is independent of k.

(b) *$\int_{\mathbf{R}} l_k(\mu, t) f(c_n - t)\, dt = \int_{\mathbf{R}} m_k(\mu, t) F_n(t)\, dt = \delta_{kn}$, where $F_n(t)$ is the Fourier transform of $f(c_n - t)$.*

(c) *For $g(t) \in L_2(\mathbf{R})$, let*

$$b_k(g) = \int_{\mathbf{R}} l_k(\mu, t) g(t)\, dt.$$

Then, for every $0 < \delta < \alpha$, there is a value of μ with $0 < \mu < 1/2b$ and a number γ such that for all real t

$$|b_n(g) f(c_n - t)| \leq c^2 \|g\|_2 \exp\left(-\delta\left(\frac{c_n^2 + |c_n|^p}{2}\right) + \gamma t^2\right),$$

where c is independent of n, and if for this value of μ, $S(g, t) = \sum b_n(g) f(c_n - t)$, then $|S(g, t)| \leq M(t)\|g\|_2$, where

$$M(t) = c \exp(\gamma t^2) \sum \exp\left(-\delta\left(\frac{c_n^2 + |c_n|^p}{2}\right)\right).$$

Using the lemma, we obtain the following representation:

THEOREM 1. *Suppose S is the linear span of* $\{f(c_n - t)\}$ *and* $g(t)$ *is in the* $L_2(\mathbf{R})$ *closure of S. Then there exists a sequence* $\{b_n\}$ *of real numbers such that*

$$g(t) = \sum b_n f(c_n - t) \qquad \text{a.e. on } \mathbf{R}.$$

The proof of Theorem 1 will be omitted since it is identical to that of Zalik [1, p. 262, Theorem 1].

Finally we obtain the following result on the degree of approximation:

THEOREM 2. *Let* $g(t)$ *be a function in the* $L_2(\mathbf{R})$ *closure of S. Let* (A, B) *be a bounded interval,* $g(t)$ *be continuous on* (A, B), *and* d_n *denote the uniform distance from* $g(t)$ *to the span of* $\{f(c_r - t)\colon\ r = 0, 1, \dots, n\}$ *in* (A, B). *Then for any* $0 < \delta < \alpha$, *there is a positive number D (independent of n and g) such that*

$$d_n \le D\|g\|_2 \exp(-\delta\rho n).$$

Proof of Lemma. We shall only prove (a) because the proofs of (b) and (c) are identical to those of Zalik [1].

We shall only consider the case in which $c_n \neq 0$ for all n, with the other case being similar. Moreover, we shall only consider the case in which p is even (the case p is odd is similar in which the last exponent below is $(1/p)(z/|c_n|^p)^p$).

Let

$$P_k(z) = \prod^{(k)}\left(1 - \frac{z}{|c_n|^p}\right) \times \exp\left\{\frac{z}{|c_n|^p} + \frac{1}{2}\left(\frac{z}{|c_n|^p}\right)^2 + \cdots + \frac{1}{p-1}\left(\frac{z}{|c_n|^p}\right)^{p-1}\right\}.$$

Then

$$P_k(z)P_k(-z) = \prod^{(k)}\left(1 - \frac{z^2}{|c_n|^{2p}}\right) \times \exp\left\{\left(\frac{z}{|c_n|^p}\right)^2 + \cdots + \frac{2}{p-2}\left(\frac{z}{|c_n|^p}\right)^{p-2}\right\} := r_k(z).$$

But for any positive number ε, an application of Luxemburg and Korevaar [2, p. 33, Lemma 7.2] with $\lambda_m = |c_m|^p$ shows that

$$\prod^{(k)} \left| 1 - \frac{|c_k|^{2p}}{|c_n|^{2p}} \right| \geq \frac{\exp(-\varepsilon)|c_k|^p}{|c_k|^p} \qquad \text{as } |c_k| \to \infty.$$

Now

$$\begin{aligned} \left| r_k\left(|c_k|^p\right) \right| &= \prod^{(k)} \left| 1 - \frac{|c_k|^{2p}}{|c_n|^{2p}} \right| \exp\left\{ \left| \frac{c_k}{c_n} \right|^{2p} + \cdots + \frac{2}{p-2} \left| \frac{c_k}{c_n} \right|^{p(p-2)} \right\} \\ &\geq \prod^{(k)} \left| 1 - \frac{|c_k|^{2p}}{|c_n|^{2p}} \right|. \end{aligned}$$

So

$$\left| r_k\left(|c_k|^p\right) \right| \geq \frac{\exp\left(-\varepsilon |c_k|^p\right)}{|c_k|^p} \qquad \text{as } |c_k| \to \infty.$$

In particular, for $\varepsilon = \mu/2$,

$$\left| r_k\left(|c_k|^p\right) \right| \geq \frac{\exp\left(-(\mu/2)|c_k|^p\right)}{|c_k|^p} \qquad \text{as } |c_k| \to \infty.$$

Thus

$$\exp\left\{ \frac{\mu}{2} |c_k|^p \right\} \left| r_k\left(|c_k|^p\right) \right| \geq |c_k|^{-p}$$

as $|c_k| \to \infty$.

Hence, there is a positive number D such that

$$\exp\left\{ \frac{\mu}{2} |c_k|^p \right\} \left| r_k\left(|c_k|^p\right) \right| \geq D. \tag{1}$$

If $n_k(r)$ denotes the number of elements in the sequence $\{c_n\colon n \neq k\}$ within the disk of radius r and $n(r)$ denotes the number of elements in the sequence $\{c_n\}$, then clearly $n_k(r) \leq n(r)$. Setting $|z| = r$ and applying to

$P_k(z)$ the same technique as in the proof of Boas [3, pp. 29–30, Lemma 2.10.13], we see that

$$\log|P_k(z)| \le Kr^{p-1}\int_0^r t^{-p} n_k(t)\,dt + Kr^p \int_r^\infty t^{-p-1} n_k(t)\,dt$$
$$\le Kr^{p-1}\int_0^r t^{-p} n(t)\,dt + Kr^p \int_r^\infty t^{-p-1} n(t)\,dt.$$

Then $\log|P_k(z)| = o(r^p)$ and similarly $\log|P_k(-z)| = o(r^p)$ uniformly for k. Thus

$$\log|P_k(z)| + \log|P_k(-z)| = o(r^p) \qquad \text{for all } k.$$

Therefore

$$\log|P_k(z)P_k(-z)| = o(r^p) \qquad \text{for all } k.$$

Then

$$\log|r_k(z)| = o(r^p) \qquad \text{for all } k.$$

So

$$\frac{\log|r_k(z)|}{r^p} = o(1) \qquad \text{for all } k.$$

Consequently, there is a function $u(r)$ such that $\lim u(r) = 0$ as $r \to \infty$, and

$$|r_k(z)| \le \exp\{u(r) r^p\} \tag{2}$$

for all k and all complex numbers z.

Set

$$q_k(\mu, z) = \exp\left\{-\frac{\mu}{2}\left(z^2 - c_k^2\right)\right\} \frac{r_k(z^p)}{r_k\left(|c_k|^p\right)}.$$

Clearly $q_k(\mu, c_n) = \delta_{kn}$ (the case p is odd is treated using the definition $r_k(z) = P_k(z)P_k(-z)$).

Now

$$\begin{aligned}
&\int_{\mathbf{R}} |q_k(\mu, x+iy)|^2\,dx \\
&\qquad = \int_{\mathbf{R}} \left|\exp\left(-\frac{\mu}{2}\left((x+iy)^2 - c_k^2\right)\right)\right|^2 \frac{\left|r_k\left((x+iy)^p\right)\right|^2}{\left|r_k\left(|c_k|^p\right)\right|^2}\,dx \\
&\qquad = \int_{\mathbf{R}} \left|\exp\left\{-\frac{\mu}{2}\left(x^2 - y^2 - c_k^2 + 2ixy\right)\right\}\right|^2 \frac{\left|r_k\left((x+iy)^p\right)\right|^2}{\left|r_k\left(|c_k|^p\right)\right|^2}\,dx \\
&\qquad = \int_{\mathbf{R}} \exp\left\{-\mu\left(x^2 - y^2 - c_k^2\right)\right\} \frac{\left|r_k\left((x+iy)^p\right)\right|^2}{\left|r_k\left(|c_k|^p\right)\right|^2}\,dx \\
&\qquad = \int_{\mathbf{R}} \exp\left\{\mu\left(y^2 + c_k^2\right)\right\}\exp\left\{-\mu x^2\right\} \frac{\left|r_k\left((x+iy)^p\right)\right|^2}{\left|r_k\left(|c_k|^p\right)\right|^2}\,dx.
\end{aligned}$$

Now using (1) and (2) an easy calculation shows that

$$\int_{\mathbf{R}} |q_k(\mu, x+iy)|^2\,dx \le d_1^2 \exp\left\{\mu\left(y^2 + |c_k|^p + c_k^2\right)\right\},$$

where d_1 is independent of k.

Similarly,

$$\begin{aligned}
&\int_{\mathbf{R}} |(x+iy)q_k(\mu, x+iy)|^2\,dx \\
&\qquad \le d_2^2 \exp\left\{\mu\left(y^2 + |c_k|^p + c_k^2\right)\right\},
\end{aligned}$$

where d_2 is independent of k.

Now, by Boas [3, p. 29, Lemma 2.10.13], $P_k(z)$ and consequently $P_k(-z)$ is of growth $(p, 0)$. Thus $r_k(z)$ is of growth $(p, 0)$. It is therefore easy to see that

$$q_k(\mu, z) = O\left(\exp\left\{-\alpha|x|^{p-1}\right\}\right)$$

as $|x| \to \infty$ on any strip of the form $|y| < \delta$.

By Titchmarch [4, p. 44, Theorem 26] applied to $q_k(\mu, t)$, it is easy to see that there is an entire function $h_k(\mu, t)$ such that $h_k(\mu, t)$ is in $L_2(\mathbf{R})$ and $q_k(\mu, z)$ is the Fourier transform of $h_k(\mu, t)$.

Moreover, $h_k(\mu, t)$ is continuous and for real values t

$$|h_k(\mu, t)| \leq d \exp\left\{-\frac{t^2}{2\mu} + \mu\left(\frac{c_k^2 + |c_k|^p}{2}\right)\right\},$$

where d is independent of k.

Let $\mu < 1/2b$ and let $m_k(\mu, t) = h_k(\mu, t)/F(t)$.

Since

$$h(t) = \frac{\exp(-bt^2)}{|F(t)|},$$

it follows that

$$|m_k(\mu, t)| = \frac{|h_k(\mu, t)|}{|F(t)|} = \frac{|h_k(\mu, t)|h(t)}{\exp(-bt^2)}$$
$$\leq d \exp\left\{-\left(\frac{1}{2\mu} - b\right)t^2 + \mu\left(\frac{c_k^2 + |c_k|^p}{2}\right)\right\} h(t),$$

where d is independent of k. Finally let $l_k(t)$ be the inverse transform of $m_k(t)$.

Proof of Theorem 2. By Theorem 1, we have $S(g, t) = g(t)$ a.e. on $\mathbf{R}$. But from part (c) of the Lemma and the continuity of $f(c_n - t)$, it follows that $S(g, t)$ is continuous on $\mathbf{R}$ and therefore $S(g, t) = g(t)$ a.e. on (A, B). Thus, $\sum b_r(g) f(c_r - t) = g(t)$ a.e. on (A, B). If $t \in (A, B)$ and $W^2 = \max\{A^2, B^2\}$, then

$$|b_r(g) f(c_r - t)| \leq c^2 \|g\|_2 \exp\left(-\delta\left(\frac{c_r^2 + |c_r|^p}{2}\right) + \gamma t^2\right)$$
$$\leq c^2 \|g\|_2 \exp\left(-\delta\left(\frac{c_r^2 + |c_r|^p}{2}\right) + \gamma W^2\right).$$

Consequently,

$$d_n \le \left| g(t) - \sum_0^n b_r(g) f(c_r - t) \right| \le \sum_{n+1}^{\infty} |b_r(g) f(c_r - t)|$$

$$\le c^2 \|g\|_2 \exp(\gamma W^2) \sum_{n+1}^{\infty} \exp\left(-\delta \left(\frac{c_r^2 + |c_r|^p}{2} \right) \right)$$

$$= c^2 \exp(\gamma W^2) \exp\left(-\frac{\delta c_0^2}{2} \right) \exp\left(\frac{-\delta |c_0|^p}{2} \right) \|g\|_2 \sum_{n+1}^{\infty} \exp\left\{ -\frac{\delta}{2} (c_r^2 - c_0^2) \right\}$$

$$\times \exp\left\{ -\frac{\delta}{2} (|c_r|^p - |c_0|^p) \right\}$$

$$\le c^2 \exp(\gamma W^2) \exp\left\{ -\frac{\delta}{2} (c_0^2 + |c_0|^p) \right\} \|g\|_2$$

$$\times \sum_{n+1}^{\infty} \exp\left\{ -\frac{\delta}{2} \rho r \right\} \exp\left\{ -\frac{\delta}{2} \rho r \right\}$$

$$= c^2 \exp(\gamma W^2) \exp\left\{ -\frac{\delta}{2} (c_0^2 + |c_0|^p) \right\} \|g\|_2 \sum_{n+1}^{\infty} \exp\{ -\delta \rho r \}$$

$$= c^2 \exp(\gamma W^2) \exp\left\{ -\frac{\delta}{2} (c_0^2 + |c_0|^p) \right\} \|g\|_2 \frac{\exp\{ -\delta \rho n \}}{\exp(\delta \rho) - 1}.$$

Let

$$D = \frac{c^2 \exp(\gamma W^2) \exp\{ -(\delta/2)(c_0^2 + |c_0|^p) \}}{\exp(\delta \rho) - 1}.$$

Note that D is a positive number (independent of n and g) and that

$$d_n \le D \|g\|_2 \exp\{ -\delta \rho n \}.$$

Remark. The above results generalize Zalik's results as one can see by taking $p = 2$.

ACKNOWLEDGMENT

This is a concise summary of a chapter in my Ph.D. dissertation. I am very grateful to Professor Richard A. Zalik, my Advisor, for his assistance and thoughtfulness.

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