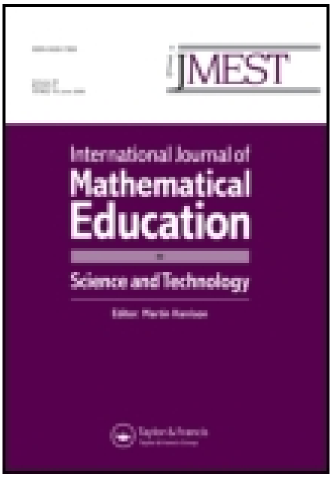

International Journal of Mathematical Education in Science and Technology
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tmes20

Maple explorations, perfect numbers, and Mersenne primes

B. Ghusayni

[a] Department of Mathematical Sciences , Faculty of Science-1, Lebanese University , Hadath, Lebanon

Published online: 20 Feb 2007.

International Journal of Mathematical Education in Science and Technology, Vol. 36, No. 6, 2005, 643–654

Maple explorations, perfect numbers, and Mersenne primes

B. GHUSAYNI*

Department of Mathematical Sciences, Faculty of Science-1,
Lebanese University, Hadath, Lebanon

(*Received 1 April 2004*)

Some examples from different areas of mathematics are explored to give a working knowledge of the computer algebra system Maple. Perfect numbers and Mersenne primes, which have fascinated people for a very long time and continue to do so, are studied using Maple and some questions are posed that still await answers.

1. Maple explorations

Maple [1] is a computer algebra system known for its ease of use and its full coverage of different subjects of the sciences. It can save its user time and frustration in calculation not to mention the fact that it gives exact values (not approximate values). The author prefers Maple to other systems but this does not come from bias but rather from experience. As a matter of fact, Maple competes in this respect with more specialized computer algebra systems like MATLAB (short for Matrix Laboratory), as the following example shows (Maple commands are preceded by > and always end with;):

> A := matrix([[1, 2], [3, 4]]);

$$A := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

> *evalm*(A∧20);

$$\begin{bmatrix} 95799031216999 & 139620104992450 \\ 209430157488675 & 305229188705674 \end{bmatrix}.$$

Notice that Maple computes exactly the 20th power of the matrix, which gives Maple superiority over MATLAB, which gives only an approximate computation.

The author, as many others, feels that the best way to learn Maple is by doing and therefore we choose a few examples to become familiar with a number of Maple

*Email: bgou@ul.edu.lb

International Journal of Mathematical Education in Science and Technology
ISSN 0020–739X print/ISSN 1464–5211 online
http://www.tandf.co.uk/journals
DOI: 10.1080/00207390500064080

commands, and we introduce each of these commands with only a brief description. The reader can then pursue this as needed.

Exact simplification:

$$> 1/6 + 1/7;$$
$$\frac{13}{42}$$

An approximation of π to 20 decimal places:

$$> evalf(Pi, 20);$$
$$3.1415926535897932385$$

(the command 'evalf' stands for 'evaluate using floating-point arithmetic' which evaluates π numerically).

Solving equations in a complex variable:

$$> solve(z \wedge 3 = 1);$$
$$1,\ -\frac{1}{2}+\frac{1}{2}\sqrt{3}I,\ -\frac{1}{2}-\frac{1}{2}\sqrt{3}I$$

The absolute value of (in this case a complex number):

$$> abs(2 + 3 * I);$$
$$\sqrt{13}$$

Division of complex numbers:

$$> (13 - 2 * I)/(2 + 3 * I);$$
$$\frac{20}{13}-\frac{43}{13}I$$

The sum of cubes of all integers from 1 to 123456789:

$$> S := sum(k \wedge 3, k = 1 \ldots 123456789);$$
$$S = 58076431640403000742495567559025$$

Factor the preceding number:

$$> ifactor(S);$$
$$(3)^4(5)^2(37)^2(3803)^2(3607)^2(333667)^2$$

(the command 'ifactor' stands for 'integer factorization').

Sum of the first 50 odd integers:

$$> sum(2 * k + 1, k = 0 \ldots 49);$$
$$2500$$

Product of the first 50 odd integers:

$$> product(2 * k + 1, k = 0 \ldots 49);$$
2725392139750729502980713245400918633290
796330545803413734328823443106201171875

Isprime function-primality testing

The function isprime is a probabilistic primality testing routine. It returns false if n is shown to be composite within one strong pseudo-primality test and one Lucas test and returns true otherwise. If isprime returns true, then n is 'very probably' prime [2, Section 4.5.4, Algorithm P]. No counter example is known and it has been conjectured that such a counter example must be hundreds of digits long.

$$> isprime(139);$$
$$true$$
$$> isprime(2317);$$
$$false$$

We can find the prime number just previous to a given composite number:

$$> prevprime(100);$$
$$97$$

We can let Maple list for us the first 50 prime numbers:

```
> s:=NULL: for i to 50 do s:=s,ithprime(i)od:s;
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61,
67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131,
137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197,
199, 211, 223, 227, 229
```

Generating Bernoulli numbers:

$$> bernoulli(4);$$
$$-\frac{1}{30}$$

Factoring expressions:

$$> E := x \wedge 6 - y \wedge 6;$$
$$E := x^6 - y^6$$
$$> factor(E);$$
$$(x-y)(x+y)(x^2+xy+y^2)(x^2-xy+y^2)$$
$$> factor(x^{12}-1);$$
$$(x-1)(x+1)(x^2+x+1)(x^2-x+1)(x^2+1)(x^4-x^2+1)$$

Infinite sums:

$$> sum(1/k \wedge 2, k = 1 \ldots infinity);$$
$$1/6\pi^2$$
$$> sum(1/k \wedge 4, k = 1 \ldots infinity);$$
$$1/90\pi^4$$
$$> sum(1/k \wedge 3, k = 1 \ldots infinity);$$
$$\zeta(3)$$

Finding limits:

$$> L := \sin(x)/x;$$
$$L := \frac{\sin(x)}{x}$$
$$> limit(L, x = 0);$$
$$1$$

Derivatives:

$$> diff(\cos(x), x);$$
$$-\sin(x)$$

Differential equations:

$$> dsolve(diff(f(x), x\$2) - f(x) = 0, f(x));$$
$$f(x) = C_1e^x + C_2e^{-x}$$

Integrals:

$$> int(1/(x \wedge 4 + 1), x);$$
$$\frac{1}{8}\sqrt{2}\ln\left(\frac{x^2 + x\sqrt{2} + 1}{x^2 - x\sqrt{2} + 1}\right) + \frac{1}{4}\sqrt{2}\arctan(x\sqrt{2} + 1) + \frac{1}{4}\sqrt{2}\arctan(x\sqrt{2} - 1)$$

Solving equations:

$$> solve(x \wedge 2 + x - 5 = 0, x);$$
$$-\frac{1}{2} + \frac{1}{2}\sqrt{21},\ -\frac{1}{2} - \frac{1}{2}\sqrt{21}$$

Finding eigenvalues:

$$> with(linalg):$$
$$> A := matrix(4, 4, [1, -2, 4, 2, -2, 1, 4, 2, 0, -2, 5, 2, -2, -2, 4, 5]);$$
$$A = \begin{bmatrix} 1 & -2 & 4 & 2 \\ -2 & 1 & 4 & 2 \\ 0 & -2 & 5 & 2 \\ -2 & -2 & 4 & 5 \end{bmatrix}$$
$$> eigenvals(A);$$
$$1, 5, 3, 3$$

Finding a basis:

$$> basis(vector[1, -2, 0, -2], vector[-2, 1, -2, -2]), vector[4, 4, 5, 4], vector[2, 2, 2, 5]);$$
$$[2, 2, 2, 5], [4, 4, 5, 4], [-2, 1, -2, -2], [1, -2, 0, -2]$$

Finding the inverse of a matrix:

$$> P := matrix(4, 4, [[1, 1, 0, 1], [1, 1, 1, 1], [1, 0, 0, 1], [1, 1, 1, 0]]);$$
$$P = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$
$$> Q := inverse(P);$$
$$Q = \begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Determinants and solutions of equations:

$> with(linalg):$

$> A := matrix(3, 3, [[3, 0, 1], [6, 1, 1], [9, 0, 7]]);$

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 6 & 1 & 1 \\ 9 & 0 & 7 \end{bmatrix}$$

$> B := matrix(3, 3, [[1, 2, 1], [3, 4, 3], [-4, -2, -4]]);$

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 3 \\ -4 & -2 & -4 \end{bmatrix}$$

$> det(A);$

12

$> det(B);$

0

$> det(A + x * I);$

$12 + 22x + 11x^2 + x^3$

$> det(A + x * B);$

$12 - 34x + 14x^2$

$> det(x * I);$

x^3

$> det(A) + det(x * I);$

$12 + x^3$

$> det(A + x * I) = det(A) + det(x * I);$

$12 + 22x + 11x^2 + x^3 = 12 + x^3$

$> eq1 := det(A + x * I) = det(A) + det(x * I);$

$eq1 := 12 + 22x + 11x^2 + x^3 = 12 + x^3$

$> sols1 := solve(eq1, x);$

$sols1 := 0, -2$

$> det(x * B);$

0

$> eq2 := det(A + x * B) = det(A) + det(x * B);$

$eq2 := 12 - 34x + 14x^2 = 12$

$> sols2 := solve(eq2, x);$

$sols2 := 0, 17/7$

$> x * det(I);$

x

$> eq3 := det(A + x * I) = det(A) + x * det(I);$

$eq3 := 12 + 22x + 11x^2 + x^3 = 12 + x$

$> sols3 := solve(eq3, x);$

$$sols3 := 0, \frac{-11}{2} - \frac{1}{2}\sqrt{37}, \frac{-11}{2} + \frac{1}{2}\sqrt{37}$$

$> eq4 := det(A + x * B) = det(A) + x * det(B);$

$$eq4 := 12 - 34x + 14x^2 = 12$$

$> sols4 := solve(eq4, x);$

$$sols4 := 0, 17/7$$

$> eq5 := det(A + x * I) = 0;$

$$eq5 := 12 + 22x + 11x^2 + x^3 = 0$$

$> sols5 := solve(eq5, x);$

$$sols5 := -1, \; -5 + \sqrt{13}, \; -5 - \sqrt{13}$$

$> P := inverse(A + I);$

$$P = \begin{bmatrix} \frac{8}{23} & 0 & \frac{-1}{23} \\ \frac{-39}{46} & \frac{1}{2} & \frac{1}{23} \\ \frac{-9}{23} & 0 & \frac{4}{23} \end{bmatrix}$$

$> eq6 := det(A + x * B) = 0;$

$$eq6 := 12 - 34x + 14x^2 = 0$$

$> sols6 := solve(eq6, x);$

$$sols6 := 2, 3/7$$

2. Results

Mersenne primes are prime numbers of the form $2^p - 1$, where p is also prime. The largest known Mersenne prime at the time of writing (the 42nd one corresponding to $p - 25964951$) has 7816230 digits. To see this, note that the number $2^{25964951} - 1$ has the same number of digits as $2^{25964951}$ because their difference is 1. Thus we only compute the number of digits of $2^{25964951}$. If a positive integer n is of the form $n = 10^x$ and with $[x]$ denoting the greatest integer function, we have $10^x \leqslant n < 10^{[x]+1}$ and therefore the number n has $[x] + 1$ digits. Thus $2^{25964951} = 10^{25964951 \log_{10} 2}$ and we have $7816229 < 25964951 \log_{10} 2 < 7816230$. Therefore the number $2^{25964951} - 1$ has 7816230 digits.

Theorem 1. $\sum 1/(2^p - 1) < \infty$, where $2^p - 1$ is prime. That is, the sum of the reciprocals of Mersenne primes is finite.

Proof. Suppose $2^p - 1$ is a prime number. Since $2^p - 1 > 2^{p-1}$ for all primes p, $\sum 1/(2^p - 1) < \sum 1/(2^{p-1}) < \sum_{n=1}^{\infty} 1/2^n = 1$.

If M_p denotes the Mersenne prime $2^p - 1$, then $\sum 1/M_p < 1$. Thus the canonical product $\prod(1 - z/M_p)$ is an entire function. Moreover, if the number of Mersenne primes was finite, then $\prod(1 - z/M_p)$ would be a real polynomial and thus of order 0. Consequently, if $\prod(1 - z/M_p)$ has a non-zero order, then the number of Mersenne primes is infinite. We shall say something about the order of $\prod(1 - z/M_p)$ in the next theorem:

Theorem 2. (see [3] for similar results). The order of $\prod(1 - z/M_p)$ is $\leqslant 1$.

Proof. Let $a > 1$. Since $M_p^a > M_p$, $\sum 1/M_p^a < \infty$. Thus $\sum 1/M_p^a < \infty$ for $a \geqslant 1$. Therefore, the smallest positive integer a for which $\sum 1/M_p^a < \infty$ is 1 and consequently the genus of the zeros of $\prod(1 - z/M_p)$ is 0. This together with the Hadamard factorization theorem shows that the genus of $\prod(1 - z/M_p)$ is also 0. So $\prod(1 - z/M_p)$ is of exponential type 0. In particular, since $\prod(1 - z/M_p)$ is of exponential type, it is of order $\leqslant 1$.

There are 41 known Mersenne primes at the time of writing. The values of the primes p such that $2^p - 1$ is a prime follow:

2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 858433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951.

We can augment our Maple explorations to include Mersenne primes. We can, for instance, invoke the help session about Mersenne primes in Maple by using the command:

> *?mersenne*

We can load up the Number Theory package by using the command:

> *with*(*numtheory*) :

Notice that the colon : is used instead of the semi-colon ; because the latter will list all the functions in number theory.

Here are some examples:

Choose 127 and test it to see if it is a prime:

> *isprime*(127);

true

Suppose now that we want to test to see if 127 is a Mersenne prime:

> *numtheory*[*mersenne*](3);

7

> *numtheory*[*mersenne*](5);

31

> *numtheory*[*mersenne*](7);

127

so it is.

We can try to find other Mersenne primes:

> *numtheory*[*mersenne*](11);

false

> *numtheory*[*mersenne*](13);

8191

> *numtheory*[*mersenne*](107);

162259276829213363391578010288127

Perfect numbers have fascinated people for a very long time and continue to do so. In this paper, we look at some of their interesting properties and mention some questions that still await answers. Each Mersenne prime generates a 'perfect' number.

A natural number is called '**perfect**' if it is equal to the sum of its positive divisors, excluding itself. Only the first four perfect numbers 6, 28, 496, and 8128 were known to the ancient Greeks; Nicomachus in his *Introductio Arithmeticae* lists them and conjectures that the nth perfect number contains exactly n digits. His conjecture is false because the next perfect number is 33550336.

We can augment our Maple explorations to include perfect numbers. Let us find the divisors of 28:

$$> d := divisors(28);$$
$$d := \{1, 2, 4, 7, 14, 28\}$$

Now let us sum the (proper) divisors of 28:

$$> sum(d['i'], 'i' = 1..5);$$
$$28$$

so 28 is perfect.

To appreciate what Maple can help us do, let us choose a much bigger number—33550336, say—and find its (proper) divisors:

$> s := divisors(33550336);$
$s := \{1, 2, 4, 8, 16, 32, 64, 128, 8191, 1024, 2048, 4096,$
$256, 16382, 32764, 65528, 524224, 262112, 1048448, 16775168,$
$8387584, 4193792, 512, 131056, 2096896, 33550336\}$

Now let us sum the (proper) divisors of 33550336:

$$> sum(s['i'], 'i' = 1 \ldots 25);$$
$$33550336$$

so 33550336 is perfect.

The first breakthrough about perfect numbers came about when Euclid proved in his *Elements*, Book IX, Proposition 36 that if $2^p - 1$ is a prime number, then $2^{p-1}(2^p - 1)$ is a perfect number. It took approximately 2000 years for the second breakthrough to come about when Euler proved that all even perfect numbers must be of the form indicated by Euclid. An old unsolved problem in mathematics is whether or not an odd perfect number exists. Brent, Cohen and Te Riele [4] have shown that there is no odd perfect number below 10^{300}. Moreover, Hagis [5] has shown (his announcement of this result [6] came a few years earlier in 1975) that an odd perfect number must have at least 8 prime factors. Furthermore, Heath-Brown [7] has shown that if n is an odd perfect number with at most k prime factors, then $n < 4^{4^k}$.

The number theoretic function $\sigma(n)$ denotes the sum of positive divisors of a natural number n (including itself). Therefore, the sum of the positive divisors of n excluding itself is $\sigma(n) - n$. It follows that a number is perfect if $\sigma(n) = 2n$. This will be used later in the paper.

Theorem 3. (Euclid). If $2^p - 1$ is a prime number, then $2^{p-1}(2^p - 1)$ is a perfect number.

Proof. If $2^p - 1 = q$ is a prime number, then the only possible divisors of $2^{p-1}(2^p - 1)$ are:

$$1 \text{ and } 2^p - 1; 2 \text{ and } 2(2^p - 1); 2^2 \text{ and } 2^2(2^p - 1); \cdots; 2^{p-1} \text{ and } 2^{p-1}(2^p - 1)$$

whose sum is

$$[1+2+2^2+\cdots+2^{p-1}][1+(2^p-1)]=(2^p-1)2^p$$

Thus the sum without $2^{p-1}(2^p-1)$ itself is

$$(2^p-1)2^p-2^{p-1}(2^p-1)=(2^p-1)2^{p-1}(2-1)=2^{p-1}(2^p-1)$$

The converse is also true:

Theorem 4. (Euler). If n is an even perfect number, then $n=2^{p-1}(2^p-1)$ for some prime number p, and 2^p-1 is also a prime number.

Proof. We can write the even number n as $n=2^{k-1}d$, where k is a natural number $\geqslant 2$ and d is an odd number. Let S denote the sum of positive divisors of d. The only positive divisors of n are the divisors of d, their doubles, ..., their multiples of 2^{k-1}. Since we know further that n is perfect, we get

$$n=2^{k-1}d=(1+2+\cdots+2^{k-1})S-n$$

or

$$2n=2^kd=(2^k-1)S$$

Thus

$$S=d+\frac{d}{2^k-1}$$

Since S and d are integers, $d/(2^k-1)$ must also be an integer. Thus $(2^k-1)/d$ and $d/(2^k-1)$ must be among the divisors of d. There can be only two divisors of d, namely d itself and $d/(2^k-1)$. But 1 is a divisor of d and hence $d=2^k-1$. Thus 2^k-1 has no other positive divisors and consequently 2^k-1 is a prime number.

Accordingly the problem of finding even perfect numbers is the same as the problem of determining primes p such that 2^p-1 is also prime.

Let M_p denote the Mersenne prime 2^p-1. By Theorem 1, $\sum 1/M_p<1$. Thus the following theorem follows immediately:

Theorem 5. The sum of the reciprocals of perfect numbers is finite.

Theorem 6. Suppose $n=2^{p-1}(2^p-1)$ is a perfect number, then $1+2+\cdots+(2^p-1)=n$. That is, an even perfect number is an arithmetic progression sum ending with the corresponding Mersenne prime.

Proof. If $S=1+2+\cdots+k$, then $S=k(k+1)/2$. Thus

$$1+2+\cdots+(2^p-1)=(2^p-1)(2^p)/2=2^{p-1}(2^p-1)$$

Theorem 7. Every even perfect number (except 6) is of the form $1^3+3^3+5^3+\cdots+(2^k-1)^3$, where k is an integer $\geqslant 2$. That is, an even perfect number (except 6) is a sum of consecutive odd cubes.

Proof. For all integers $k\geqslant 2$ we have:

$$\begin{aligned}
&1^3+3^3+5^3+\cdots+(2^k-1)^3\\
&\quad=1^3+2^3+3^3+\cdots+(2^k-2)^3+(2^k-1)^3-[2^3+4^3+\cdots+(2^k-2)^3]\\
&\quad=[2^{k-1}(2^k-1)]^2-2^3[2^{k-2}(2^{k-1}-1)]^2\\
&\quad=[2^{k-1}(2^k-1)]^2-2[2^{k-1}(2^{k-1}-1)]^2\\
&\quad=2^{2k-2}[(2^k-1)^2-2(2^{k-1}-1)^2]
\end{aligned}$$

$$= 2^{2k-2}[2^{2k} - 2^{2k-1} - 1]$$
$$= 2^{2k-2}(2.2^{2k-1} - 2^{2k-1} - 1)$$
$$= 2^{2k-2}(2^{2k-1} - 1)$$

Now let n be any even perfect number different from 6. Then $n = 2^{p-1}(2^p - 1)$ for some odd prime p. Letting $k = (p+1)/2$ we get $1^3 + 3^3 + \cdots + (2^{(p+1)/2} - 1)^3 = 2^{p-1}[2^p - 1] = n$ which is the desired result.

Theorem 8. The sum of the reciprocals of all positive divisors of a perfect number is always 2.

Proof. Let n be a perfect number. We will show that $\sum 1/d = 2$ over the positive divisors d of n. Clearly $\sigma(n) = \sigma(n/d)$. Thus $(1/n)\sigma(n) = \sum 1/d$. Consequently, $\sum 1/d = 2n/2 = 2$. This completes the proof.

Theorem 9. [8] 28 is the only perfect number of the form $n^n + 1$, where n is a positive integer.

Theorem 10. All even perfect numbers end in 6 or 8.

Proof. It is well known that every integer is congruent (mod m) to one of $0, 1, \ldots, m-1$. In particular, an integer is congruent(mod 10) to one of $0, 1, \ldots, 9$ and moreover an even integer is congruent (mod 10) to one of $1, 3, 5, 7, 9$. Now an even perfect number $n = 2^{p-1}(2^p - 1)$ has the following possibilities for p: $p = 2, p = 5$, and p congruent to $1, 3, 7$, or 9 (mod 10) and these cases can easily be shown to give an even perfect number ending in $6, 6, 6, 6, 6$, and 8 respectively.

Remark. It is worth mentioning that the previous theorem cannot be pushed further to state that even perfect numbers should alternate between 6 and 8 because (like the fifth perfect number) the sixth one is $2^{17-1}(2^{17} - 1) = 8589869056$ which also ends with a 6.

Theorem 11. An even perfect number n (except 6) is congruent to 1 (*mod* 9).

Proof. By Theorem 4, $n = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is prime.

Case 1. $p = 3$: Clearly $n = 28$ is congruent to 1 (*mod* 9).

Case 2. $p \geqslant 5$ (with $2^p - 1$ prime): As in the proof of Theorem 11, we see that p is congruent to 1 or 5 (mod 6). Due to the similarity in the proof of the two cases, we only consider the case $p = 1 + 6k$, for some natural number k. Thus $2^{p-1} = 2^{6k} = (2^6)^k$ which is congruent to 1 (mod 9), say $2^{p-1} = 1 + 9L$ for some natural number L. So $2^p - 1$ is also congruent to 1 (mod 9) and consequently n is congruent to 1 (mod 9).

Theorem 12. An even perfect number n (except 6) is congruent to 4 (*mod* 6).

Proof. By Theorem 4, $n = 2^{p-1}(2^p - 1)$ where $2^p - 1$ is prime. As in the proof of Theorem 11, p is congruent to 1 or 5 (mod 6). In both cases it is easy to see that 2^{p-1} is congruent to 1 (mod 3). Thus for odd prime numbers p there are odd natural numbers k such that $2^{p-1} = 1 + 3k$. Therefore $n = 1 + 9k + 18k^2 = 4 - 3 + 9k + 18k^2 = 4 + 3(6k^2 + 3k - 1)$. Since k is odd, it is clear that $6k^2$ and $3k - 1$ are even.

Theorem 13. [9, corollary, p. 232]. If an odd perfect number exists, then it is congruent to 1 (mod 4).

Theorem 14. The pair 5 and 7 is the only pair of twin primes (primes that differ by 2) whose sum divided by 2 is a perfect number.

Proof. Clearly the pair 5 and 7 satisfies the above property.
If p is a prime > 5 with $p+2$ prime, then p is congruent to 5 (mod 6) (clearly, the case p congruent to 1 (mod 6) is ruled out here for otherwise $p+2$ would not be a prime number). That is, $p = 5 + 6k$ for some natural number k. Then m is a natural number > 1, with $(p+(p+2))/2 = p + 1 = 6m$ which is not a perfect number.

3. Unsolved problems with comments

(1) Are there infinitely many perfect numbers? or equivalently:
Are there infinitely many Mersenne prime numbers?

From the proof of Theorem 2 we see that $\prod(1 - z/M_p)$ has genus 0 which, in turn, implies that $\prod(1 - z/M_p)$ is of order $\leqslant 1$. Can one use the fact that $\prod(1 - z/M_p)$ is of genus 0 more effectively to obtain a sharper result on the order of $\prod(1 - z/M_p)$? More precisely, if one could show that the order of $\prod(1 - z/M_p)$ is non-zero, that would imply that there are infinitely many Mersenne prime numbers.

(2) Is there an odd perfect number?

Ore [9] considered the harmonic mean $H(n)$ of a natural number n defined by $1/H(n) = 1/\nu(n)\cdot\sum(1/d)$ over the positive divisors d of n with $\nu(n)$ denoting the numbers of such divisors. Now

$$H(n) = \frac{\nu(n)}{\sum(1/d)} = \frac{n\nu(n)}{\sigma(n)}.$$

Thus for a perfect number n, $H(n)$ is an integer (clearly here $H(n) = \nu(n)/2$): if n is even, then $n = 2^{p-1}(2^p - 1)$ has $\nu(n) = 2p$ and thus $H(n) = p$ is an integer. If n is odd, then by a corollary [10, p. 232] of another result of Euler we have $n = p^t k^2$, where p is a prime not dividing k and p and t are congruent to 1 (mod 4); that is $n = p^{4g+1}k^2$ for some integer g and p an odd prime. So $\nu(n)$ is also even in this case because the p-exponent is odd.

Based on some numerical computations [9], Ore conjectures that a number with integral harmonic mean of divisors must be even. If Ore's Conjecture is true, then that would imply that there are no odd perfect numbers (Ore verified his conjecture for $n < 10^4$. Mills [11] verified Ore's Conjecture for $n < 10^7$ as well as when all prime-power factors of n are $< 65551^2$. Finally, Pomerance (see for example [12, p. 84]) showed that if the number of distinct prime factors of a perfect number n is at most 2 and $H(n)$ is an integer, then n is even).

References

[1] Char, B.W., Geddes, K.O., Gonnet, G.H., Leong, B.L., Monagan, M.B. and Watt, S.M., 1991, *Maple V Language Reference Manual* (New York: Springer-Verlag).

[2] Donald E. Knuth, 1981, *The Art of Computer Programming*, 2nd edn (Reading, MA: Addison-Wesley).

[3] Ghusayni, B., 1997, Products and sums with applications, *Missouri Journal of Mathematical Sciences*, **9**, 90–94.
[4] Brent, R.P., Cohen, G.L. and Te Riele, H.J.J., 1991, Improved techniques for odd perfect numbers, *Mathematics of Computation*, **57**, 857–868.
[5] Hagis, P., 1980, Outline of a proof that every odd perfect number has at least 8 prime factors, *Mathematics of Computation*, **35**, 1027–1032.
[6] Hagis, P., 1975, Every odd perfect number has at least eight prime factors, *Notices, American Mathematical Society*, **22**, A60.
[7] Heath-Brown, D.R., 1994, Odd perfect numbers, *Mathematical Proceedings of the Cambridge Philosophical Society*, **115**, 191–195.
[8] Iannucci, D. and Cohen, G.L., 1995, Solution to problem 570, *The American Mathematical Monthly*, **102**, 72.
[9] Ore, O., 1948, On the averages of the divisors of a number, *The American Mathematical Monthly*, **55**, 915–619.
[10] Burton, D., 1980, *Elementary Number Theory* (Boston: Allyn and Bacon).
[11] Mills, W.H., 1972, *On a conjecture of Ore*, Proceedings of Number Theory Conference in Boulder, 142–146.
[12] Ribenboim, P., 1989, *The Book of Prime Number Records*, 2nd edn (New York: Springer-Verlag).